

CONTINUITY AND COMPACTNESS FOR PSEUDO-DIFFERENTIAL OPERATORS WITH SYMBOLS IN QUASI-BANACH SPACES OR HÖRMANDER CLASSES

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ABSTRACT. We deduce continuity and Schatten-von Neumann properties for operators with matrices satisfying mixed quasi-norm estimates with Lebesgue and Schatten parameters in $(0, \infty]$. We use these results to deduce continuity and Schatten-von Neumann properties for pseudo-differential operators with symbols in quasi-Banach modulation spaces, or in appropriate Hörmander classes.

0. INTRODUCTION

The singular values for a linear operator is a non-increasing sequence of non-negative numbers which are strongly linked to questions on continuity and compactness for the operator in the following sense:

- the operator is continuous, if and only if its singular values are bounded.
- the operator is compact, if and only if its singular values decay towards zero at infinity. Moreover, fast decays of the singular values permit efficient finite rank approximations.
- the operator has rank $j \geq 0$, if and only if its singular values of order $j + 1$ and higher are zero.

(See [21, 31] and Section 1 for definitions.)

In particular, there is a strong connection between the decay of the singular values and finding pseudo-inverses in convenient ways, since such questions are linked to efficient finite rank approximations.

One way to measure the decay of singular values is to consider Schatten-von Neumann classes. More precisely, let T be a linear operator. Then T belongs to \mathcal{S}_p , the set of Schatten-von Neumann operators of order $p \in (0, \infty]$, if and only if its singular values $\sigma_1(T), \sigma_2(T), \dots$ belong to ℓ^p . Since the singular values are non-negative and non-increasing,

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it follows that

$$\begin{aligned}\sigma_j(T) &= o(j^{-1/p}), \quad \text{when } T \in \mathcal{J}_p, \, p < \infty, \\ \sigma_j(T) &\neq o(j^{-1/p}), \quad \text{when } T \notin \mathcal{J}_{p+\varepsilon}, \, p < \infty, \, \varepsilon > 0,\end{aligned}\tag{0.1}$$

which indicates the link between Schatten-von Neumann classes and the decays of singular values.

It is in general a difficult task to find exact and convenient characterizations of Schatten-von Neumann classes. One is therefore left to find suitable necessary or sufficient conditions when characterizing such classes. For example a Toeplitz operator, acting on L^2 belongs to \mathcal{J}_p , $p \in [1, \infty]$, when its symbol belongs to L^p (cf. [1, 2, 29]). For pseudo-differential operators $\text{Op}(a)$, the situation is slightly different since $\text{Op}(a)$ might not be in \mathcal{J}_p , $p \neq 2$, when its symbol a belongs to L^p . On the other hand, by adding further restrictions on the symbols it is possible to deduce similar sufficient conditions as for Toeplitz operators. For example, if $S(m, g)$ is an appropriate Hörmander class parameterized with the Riemannian metric g and weight function m on the phase space, then

$$\{ \text{Op}_t(a) ; a \in S(m, g) \} \subseteq \mathcal{J}_p \iff m \in L^p. \tag{0.2}$$

(Cf. Theorems 2.1 and 2.9 in [8]. See also [30, 31, 47] for pre-results.)

There are several Schatten-von Neumann results for pseudo-differential operators with symbols in modulation spaces, Besov spaces and Sobolev spaces (cf. [48] and the references therein). In particular, let $M^{p,q}$ be the classical modulation space with parameters $p, q \in [1, \infty]$, introduced by Feichtinger in [16]. Then

$$\text{Op}(a) \in \mathcal{J}_p \quad \text{when } a \in M^{p,p}, \, p \in [1, 2], \tag{0.3}$$

$$\text{Op}(a) : M^{p_1, q_1} \rightarrow M^{p_1, q_1} \quad \text{when } a \in M^{\infty, 1}, \, p_1, q_1 \in [1, \infty], \tag{0.4}$$

and

$$\text{Op}(a) : M^{\infty, \infty} \rightarrow M^{1, 1} \quad \text{when } a \in M^{1, 1}. \tag{0.5}$$

The relation (0.3) was essentially deduced by Gröchenig and Heil, although it seems to be well-known earlier by Feichtinger (cf. [25, Proposition 4.1]). The relation (0.4) was first proved in [21], with certain pre-results given already in [25, 42], and (0.5) is in some sense obtained by Feichtinger already in [16].

There are also several extensions and modifications of these results. For example, in [26, 45] it was proved that

$$\begin{aligned}\text{Op}(a) : M^{p_1, q_1} &\rightarrow M^{p_2, q_2} \quad \text{when } a \in M^{p, q}, \, q \leq \min(p, p') \\ \text{and } \frac{1}{p_1} - \frac{1}{p_2} &= \frac{1}{q_1} - \frac{1}{q_2} = 1 - \frac{1}{p} - \frac{1}{q}, \, q \leq p_2, q_2 \leq p,\end{aligned}\tag{0.6}$$

which covers both (0.4) and (0.5). See also [46, 48–50] for extensions of the latter result to weighted spaces, and [35, 52] for related results with other types of modulation spaces as symbol classes. Furthermore, in [13–15, 18], related analysis in background of compact or local-compact Lie groups can be found.

In the literature, it is usually assumed that p and q here above belong to $[1, \infty]$ instead of the larger interval $(0, \infty]$. An important reason for excluding the cases $p < 1$ or $q < 1$ is that the involved spaces fail to be local convex, leading in general to several additional difficulties compared to the situation when $p, q \in [1, \infty]$. On the other hand, in view of (0.1) it is valuable to decide whether an operator belongs to \mathcal{S}_p or not, also in the case $p < 1$. Here we remark that convenient Schatten- p results with $p < 1$ can be found for Hankel and Toeplitz operators in e. g. [32], and for pseudo-differential operators on compact Lie groups in e. g. [13–15].

In the paper we deduce weighted extensions of (0.2)–(0.5), where in contrast to [46, 48–50], the case $p < 1$ is included. First we deduce continuity and Schatten-von Neumann properties for suitable types of matrix operators. Thereafter we carry over these results to the case of pseudo-differential operators with symbols in modulation spaces, using Gabor analysis as link, in analogous ways as in e. g. [22, 23, 25, 27, 48, 53].

Here we remark that our analysis is comprehensive compared to [22, 23, 25, 27, 48, 53] because of the absent of local-convexity. The situation is handled by using the Gabor analysis in [19, 51], for non-local convex modulation spaces, in combination of suitable factorization techniques for matrix operators.

In order to shed some more light we explain some consequences of our investigations. As a special case of Theorem 3.4 we have

$$\text{Op}(a) \in \mathcal{S}_p \quad \text{when} \quad a \in M^{p,p}, \quad p \in (0, 2], \quad (0.3)'$$

i. e. (0.3) still holds after $[1, 2]$ is replaced by the larger interval $(0, 2]$. Furthermore, we prove that (0.3)' is sharp in the sense that any modulation space (with trivial weight), and not contained in $M^{p,p}(\mathbf{R}^{2d})$, contains symbols, whose corresponding pseudo-differential operators fail to belong to \mathcal{S}_p (cf. Theorem 3.6).

In Section 3 we also deduce general continuity results for pseudo-differential operators with symbols in modulation spaces. In particular, (0.6) is extended in Theorem 3.1 extend in several ways, and as special case, (0.4) and (0.5) are extended into

$$\begin{aligned} \text{Op}(a) : M^{p_1, q_1} &\rightarrow M^{p_1, q_1} \\ \text{when} \quad a &\in M^{\infty, q}, \quad p_1, q_1 \in [q, \infty], \quad q \in (0, 1], \end{aligned} \quad (0.4)'$$

and

$$\text{Op}(a) : M^{\infty, \infty} \rightarrow M^{q, q} \quad \text{when} \quad a \in M^{q, q}, \quad q \in (0, 1]. \quad (0.5)'$$

In Section 4 we apply (0.3)' to deduce Schatten-von Neumann properties for pseudo-differential operators with symbols in $S(m, g)$ in Hörmander-Weyl calculus. In particular we show that the sufficiency part of (0.2) still holds for $p \in (0, 1]$ (cf. Theorem 4.1). That is, for suitable m and g , we have

$$\{\text{Op}_t(a); a \in S(m, g)\} \subseteq \mathcal{J}_p \quad \text{when} \quad m \in L^p.$$

An important part behind the analysis concerns Theorem 2.1, which in the non-weighted case, essentially state that any matrix $A \in \mathbb{U}^{p_0}$ can be factorized as

$$A = A_1 \cdot A_2, \quad \text{when} \quad A_j \in \mathbb{U}^{p_j}, \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_0}. \quad (0.7)$$

From these relations we obtain

$$\mathbb{U}^p \subseteq \mathcal{J}_p, \quad \text{when} \quad p \in (0, 2]. \quad (0.8)$$

In fact, the set of Hilbert-Schmidt operators on ℓ^2 agrees with \mathbb{U}^2 , and with \mathcal{J}_2 (also in norms). Consequently, $\mathbb{U}^2 = \mathcal{J}_2$, and Hölder's inequality for Schatten-von Neumann classes together with (0.7) give that for every $A \in \mathbb{U}^{2/N}$, with integer $N \geq 1$, there are matrices $A_1, \dots, A_N \in \mathbb{U}^2$ such that

$$A = A_1 \cdots A_N \in \mathbb{U}^2 \circ \cdots \circ \mathbb{U}^2 = \mathcal{J}_2 \circ \cdots \circ \mathcal{J}_2 = \mathcal{J}_{2/N}.$$

Hence $\mathbb{U}^{2/N} \subseteq \mathcal{J}_{2/N}$ for every integer $N \geq 1$. A (real) interpolation argument between the cases

$$\mathbb{U}^{2/N} \subseteq \mathcal{J}_{2/N} \quad \text{and} \quad \mathbb{U}^2 = \mathcal{J}_2$$

now shows that $\mathbb{U}^p \subseteq \mathcal{J}_p$ when $p \in [2/N, 2]$. Since $2/N$ can be chosen arbitrarily close to 0, (0.8) follows.

In Section 2, the previous arguments are used to deduce more general versions of (0.8) involving weighted spaces. (See Theorem 2.5.)

In Section 5 we show some applications and other results for Schatten-von Neumann symbols. Here we introduce the set $s_{t,p}^q$ consisting of all symbols a such that $\text{Op}_t(a)$ belongs to \mathcal{J}_p and such that the orthonormal sequences of the eigenfunctions to $|\text{Op}_t(a)|$ and $|\text{Op}_t(a)^*|$ are bounded sets in the modulation space M^{2q} . It follows that $s_{t,p}^q$ is contained in $s_{t,p}$ the set of all symbols a such that $\text{Op}_t(a) \in \mathcal{J}_p$.

We prove that \mathcal{S} is continuously embedded in $s_{t,p}^q$, and that

$$s_{t,p}^p \cap \mathcal{E}' \subseteq \mathcal{F}L^p \cap \mathcal{E}' \subseteq s_{t,p} \cap \mathcal{E}'$$

for every $p > 0$.

Finally we remark that in [13–15, 18], Delgado, Fischer, Ruzhansky and Turunen deal with various kinds of continuity and compactness questions for pseudo-differential operators acting on functions defined on suitable Lie groups. In their approach, matrix-valued symbols appear naturally, and several interesting results on matrices are deduced. A part of these investigations are related to the analysis in Section 2.

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1. PRELIMINARIES

In this section we recall some facts on Gelfand-Shilov spaces, modulation spaces and Schatten-von Neumann classes. The proofs are in general omitted.

1.1. Weight functions. We start by discussing general properties on weight functions. A *weight* on \mathbf{R}^d is a positive function $\omega \in L_{loc}^\infty(\mathbf{R}^d)$ such that $1/\omega \in L_{loc}^\infty(\mathbf{R}^d)$. We usually assume that ω is *moderate*, or *v-moderate* for some positive function $v \in L_{loc}^\infty(\mathbf{R}^d)$. This means that

$$\omega(x+y) \lesssim \omega(x)v(y), \quad x, y \in \mathbf{R}^d. \quad (1.1)$$

Here $A \lesssim B$ means that $A \leq cB$ for a suitable constant $c > 0$, and for future references, we write $A \asymp B$ when $A \lesssim B$ and $B \lesssim A$. We note that (1.1) implies that ω fulfills the estimates

$$v(-x)^{-1} \lesssim \omega(x) \lesssim v(x), \quad x \in \mathbf{R}^d. \quad (1.2)$$

Furthermore, if v in (1.1) can be chosen as a polynomial, then ω is called a weight of *polynomial type*. We let $\mathcal{P}(\mathbf{R}^d)$ and $\mathcal{P}_E(\mathbf{R}^d)$ be the sets of all weights of polynomial type and moderate weights on \mathbf{R}^d , respectively.

It can be proved that if $\omega \in \mathcal{P}_E(\mathbf{R}^d)$, then ω is *v-moderate* for some $v(x) = e^{r|x|}$, provided the positive constant $r > 0$ is chosen large enough (cf. [24]). In particular, (1.2) shows that for any $\omega \in \mathcal{P}_E(\mathbf{R}^d)$, there is a constant $r > 0$ such that

$$e^{-r|x|} \lesssim \omega(x) \lesssim e^{r|x|}, \quad x \in \mathbf{R}^d \quad (1.3)$$

(cf. [24]).

We say that v is *submultiplicative* if v is even and (1.1) holds with $\omega = v$. In the sequel, v and v_j for $j \geq 0$, always stand for submultiplicative weights if nothing else is stated.

1.2. Gelfand-Shilov spaces. Next we recall the definition of Gelfand-Shilov spaces.

Let $h, s \in \mathbf{R}_+$ be fixed. Then $\mathcal{S}_{s,h}(\mathbf{R}^d)$ is the set of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{s,h}} \equiv \sup \frac{|x^\beta \partial^\alpha f(x)|}{h^{|\alpha+\beta|} (\alpha! \beta!)^s}$$

is finite. Here the supremum is taken over all $\alpha, \beta \in \mathbf{N}^d$ and $x \in \mathbf{R}^d$.

Obviously $\mathcal{S}_{s,h}(\mathbf{R}^d)$ is a Banach space which increases as h and s increase, and is contained in $\mathcal{S}(\mathbf{R}^d)$, the set of Schwartz functions on \mathbf{R}^d . Furthermore, if $s > 1/2$, or $s = 1/2$ and h is sufficiently large,

then is dense in \mathcal{S} . Hence, the dual $(\mathcal{S}_{s,h})'(\mathbf{R}^d)$ of $\mathcal{S}_{s,h}(\mathbf{R}^d)$ is a Banach space which contains $\mathcal{S}'(\mathbf{R}^d)$.

The *Gelfand-Shilov spaces* $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$ are the inductive and projective limits respectively of $\mathcal{S}_{s,h}(\mathbf{R}^d)$ with respect to h . This implies that

$$\mathcal{S}_s(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d) \quad \text{and} \quad \Sigma_s(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}(\mathbf{R}^d), \quad (1.4)$$

and that the topology for $\mathcal{S}_s(\mathbf{R}^d)$ is the strongest possible one such that each inclusion map from $\mathcal{S}_{s,h}(\mathbf{R}^d)$ to $\mathcal{S}_s(\mathbf{R}^d)$ is continuous. The space $\Sigma_s(\mathbf{R}^d)$ is a Fréchet space with semi norms $\|\cdot\|_{\mathcal{S}_{s,h}}$, $h > 0$. Moreover, $\mathcal{S}_s(\mathbf{R}^d) \neq \{0\}$, if and only if $s \geq 1/2$, and $\Sigma_s(\mathbf{R}^d) \neq \{0\}$, if and only if $s > 1/2$.

For every $\varepsilon > 0$ and $s > 0$, we have

$$\Sigma_s(\mathbf{R}^d) \subseteq \mathcal{S}_s(\mathbf{R}^d) \subseteq \Sigma_{s+\varepsilon}(\mathbf{R}^d).$$

The *Gelfand-Shilov distribution spaces* $\mathcal{S}'_s(\mathbf{R}^d)$ and $\Sigma'_s(\mathbf{R}^d)$ are the projective and inductive limit respectively of $\mathcal{S}'_{s,h}(\mathbf{R}^d)$. Hence

$$\mathcal{S}'_s(\mathbf{R}^d) = \bigcap_{h>0} \mathcal{S}'_{s,h}(\mathbf{R}^d) \quad \text{and} \quad \Sigma'_s(\mathbf{R}^d) = \bigcup_{h>0} \mathcal{S}'_{s,h}(\mathbf{R}^d). \quad (1.4)'$$

By [39], \mathcal{S}'_s and Σ'_s are the duals of \mathcal{S}_s and Σ_s , respectively.

The Gelfand-Shilov spaces are invariant or posses convenient mapping properties under several basic transformations. For example they are invariant under translations, dilations, and under (partial) Fourier transformations.

From now on we let \mathcal{F} be the Fourier transform, given by

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) e^{-i\langle x, \xi \rangle} dx$$

when $f \in L^1(\mathbf{R}^d)$. Here $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on \mathbf{R}^d . The map \mathcal{F} extends uniquely to homeomorphisms on $\mathcal{S}'(\mathbf{R}^d)$, $\mathcal{S}'_s(\mathbf{R}^d)$ and $\Sigma'_s(\mathbf{R}^d)$, and restricts to homeomorphisms on $\mathcal{S}(\mathbf{R}^d)$, $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$, and to a unitary operator on $L^2(\mathbf{R}^d)$.

Next we recall some mapping properties of Gelfand-Shilov spaces under short-time Fourier transforms. Let $\phi \in \mathcal{S}(\mathbf{R}^d)$ be fixed. For every $f \in \mathcal{S}'(\mathbf{R}^d)$, the *short-time Fourier transform* $V_\phi f$ is the distribution on \mathbf{R}^{2d} defined by the formula

$$(V_\phi f)(x, \xi) = \mathcal{F}(f \overline{\phi(\cdot - x)})(\xi) = (f, \phi(\cdot - x) e^{i\langle \cdot, \xi \rangle}). \quad (1.5)$$

We recall that if $T(f, \phi) \equiv V_\phi f$ when $f, \phi \in \mathcal{S}_{1/2}(\mathbf{R}^d)$, then T is uniquely extendable to sequentially continuous mappings

$$T : \mathcal{S}'_s(\mathbf{R}^d) \times \mathcal{S}_s(\mathbf{R}^d) \rightarrow \mathcal{S}'_s(\mathbf{R}^{2d}) \bigcap C^\infty(\mathbf{R}^{2d}),$$

$$T : \mathcal{S}'_s(\mathbf{R}^d) \times \mathcal{S}'_s(\mathbf{R}^d) \rightarrow \mathcal{S}'_s(\mathbf{R}^{2d}),$$

and similarly when \mathcal{S}_s and \mathcal{S}'_s are replaced by Σ_s and Σ'_s , respectively, or by \mathcal{S} and \mathcal{S}' , respectively (cf. [12, 49]). We also note that $V_\phi f$ takes the form

$$V_\phi f(x, \xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(y) \overline{\phi(y-x)} e^{-i\langle y, \xi \rangle} dy \quad (1.5)'$$

when $f \in L^p_{(\omega)}(\mathbf{R}^d)$ for some $\omega \in \mathcal{P}_E(\mathbf{R}^d)$, $\phi \in \Sigma_1(\mathbf{R}^d)$ and $p \geq 1$. Here $L^p_{(\omega)}(\mathbf{R}^d)$, when $p \in (0, \infty]$ and $\omega \in \mathcal{P}_E(\mathbf{R}^d)$, is the set of all $f \in L^p_{loc}(\mathbf{R}^d)$ such that $f \cdot \omega \in L^p(\mathbf{R}^d)$.

1.3. Mixed quasi-normed space of Lebesgue types. Let $p, q \in (0, \infty]$, and let $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$. Then $L^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ and $L^{p,q}_{*,(\omega)}(\mathbf{R}^{2d})$ consist of all measurable functions F on \mathbf{R}^{2d} such that

$$\|g_1\|_{L^q(\mathbf{R}^d)} < \infty, \quad \text{where} \quad g_1(\xi) \equiv \|F(\cdot, \xi)\omega(\cdot, \xi)\|_{L^p(\mathbf{R}^d)}$$

and

$$\|g_2\|_{L^p(\mathbf{R}^d)} < \infty, \quad \text{where} \quad g_2(x) \equiv \|F(x, \cdot)\omega(x, \cdot)\|_{L^q(\mathbf{R}^d)},$$

respectively.

More generally, let

$$\mathbf{p} = (p_1, \dots, p_d) \in (0, \infty]^d, \quad \mathbf{q} = (q_1, \dots, q_d) \in (0, \infty]^d,$$

S_d be the set of permutations on $\{1, \dots, d\}$, $\mathbf{p} \in (0, \infty]^d$, $\omega \in \mathcal{P}_E(\mathbf{R}^d)$, and let $\sigma \in S_d$. Moreover, let $\Omega_j \subseteq \mathbf{R}$ be Borel-sets, μ_j be positive Borel measures on Ω_j , $j = 1, \dots, d$, and let $\Omega = \Omega_1 \times \dots \times \Omega_d$ and $\mu = \mu_1 \otimes \dots \otimes \mu_d$. For every measurable and complex-valued function f on Ω , let $g_{j,\omega,\mu}$, $j = 1, \dots, d-1$, be defined inductively by

$$g_{0,\omega,\mu}(x_1, \dots, x_d) \equiv |f(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(d)})\omega(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(d)})|,$$

$$g_{k,\omega,\mu}(x_{k+1}, \dots, x_d) \equiv \|g_{k-1,\omega,\mu}(\cdot, x_{k+1}, \dots, x_d)\|_{L^{p_k}(\mu_k)}, \quad k = 1, \dots, d-1,$$

and let

$$\|f\|_{L^{\mathbf{p}}_{\sigma,(\omega)}(\mu)} \equiv \|g_{d-1,\omega,\mu}\|_{L^{p_d}(\mu_d)}.$$

The mixed quasi-norm space $L^{\mathbf{p}}_{\sigma,(\omega)}(\mu)$ of Lebesgue type is defined as the set of all μ -measurable functions f such that $\|f\|_{L^{\mathbf{p}}_{\sigma,(\omega)}(\mu)} < \infty$.

In the sequel we have $\Omega = \mathbf{R}^d$ and $d\mu = dx$, or $\Omega = \Lambda$ and $\mu(j) = 1$ when $j \in \Lambda$, where

$$\Lambda = \Lambda_{[\theta]} = T_\theta \mathbf{Z}^d \equiv \{(\theta_1 j_1, \dots, \theta_d j_d); (j_1, \dots, j_d) \in \mathbf{Z}^d\}, \quad (1.6)$$

$$\theta = (\theta_1, \dots, \theta_d) \in \mathbf{R}_*^d,$$

and T_θ denotes the diagonal matrix with diagonal elements $\theta_1, \dots, \theta_d$. In the former case we set $L^{\mathbf{p}}_{\sigma,(\omega)}(\mu) = L^{\mathbf{p}}_{\sigma,(\omega)} = L^{\mathbf{p}}_{\sigma,(\omega)}(\mathbf{R}^d)$, and in the latter case we set $L^{\mathbf{p}}_{\sigma,(\omega)}(\mu) = \ell^{\mathbf{p}}_{\sigma,(\omega)}(\Lambda)$.

For conveniency we also set $L_{(\omega)}^{\mathbf{p}} = L_{\sigma,(\omega)}^{\mathbf{p}}$ and $\ell_{(\omega)}^{\mathbf{p}} = \ell_{\sigma,(\omega)}^{\mathbf{p}}$ when σ is the identity map, and we let $\ell(\Lambda)$ be the set of all (complex-valued) sequences on Λ and $\ell_0(\Lambda)$ be the set of all $f \in \ell(\Lambda)$ such that $f(j) \neq 0$ for at most finite numbers of j . Furthermore, if ω is equal to 1, then we set

$$L_{\sigma}^{\mathbf{p}} = L_{\sigma,(\omega)}^{\mathbf{p}}, \quad \ell_{\sigma}^{\mathbf{p}} = \ell_{\sigma,(\omega)}^{\mathbf{p}}, \quad L^{\mathbf{p}} = L_{(\omega)}^{\mathbf{p}} \quad \text{and} \quad \ell^{\mathbf{p}} = \ell_{(\omega)}^{\mathbf{p}}.$$

Let $\mathbf{p} = (p_1, \dots, p_d) \in [0, \infty]^d$, $\mathbf{q} = (q_1, \dots, q_d) \in [0, \infty]^d$ and $t \in [-\infty, \infty]$. Then we use the conventions

$$\mathbf{p} \leq \mathbf{q} \quad \text{and} \quad \mathbf{p} \leq t \quad \text{when} \quad p_j \leq q_j \quad \text{and} \quad p_j \leq t,$$

respectively, for every $j = 1, \dots, d$, and

$$\mathbf{p} = \mathbf{q} \quad \text{and} \quad \mathbf{p} = t \quad \text{when} \quad p_j = q_j \quad \text{and} \quad p_j = t,$$

respectively, for every $j = 1, \dots, d$. The relations $\mathbf{p} < \mathbf{q}$ and $\mathbf{p} < t$ are defined analogously. We also let

$$\mathbf{p} \pm \mathbf{q} = (p_1 \pm q_1, \dots, p_d \pm q_d) \quad \text{and} \quad \mathbf{p} \pm t = (p_1 \pm t, \dots, p_d \pm t),$$

provided the right-hand sides are well-defined and belongs to $[-\infty, \infty]^d$. Moreover, we set $1/0 = \infty$, $1/\infty = 0$ and $1/\mathbf{p} = (1/p_1, \dots, 1/p_d)$.

We also let

$$\max(\mathbf{p}) \equiv \max(p_1, \dots, p_d) \quad \text{and} \quad \min(\mathbf{p}) \equiv \min(p_1, \dots, p_d),$$

and note that if $\max(\mathbf{p}) < \infty$, then $\ell_0(\Lambda)$ is dense in $\ell_{\sigma,(\omega)}^{\mathbf{p}}(\Lambda)$.

1.4. Modulation spaces. Next we define modulation spaces. Let $\phi \in \mathcal{S}_{1/2}(\mathbf{R}^d) \setminus 0$. For any $p, q \in (0, \infty]$ and $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$, the modulation spaces $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ and $W_{(\omega)}^{p,q}(\mathbf{R}^d)$ are the sets of all $f \in \mathcal{S}'_{1/2}(\mathbf{R}^d)$ such that $V_{\phi}f \in L_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ and $V_{\phi}f \in L_{*,(\omega)}^{p,q}(\mathbf{R}^{2d})$, respectively. We equip these spaces with the quasi-norms

$$\|f\|_{M_{(\omega)}^{p,q}} \equiv \|V_{\phi}f\|_{L_{(\omega)}^{p,q}} \quad \text{and} \quad \|f\|_{W_{(\omega)}^{p,q}} \equiv \|V_{\phi}f\|_{L_{*,(\omega)}^{p,q}},$$

respectively. One of the most common types of modulation spaces concerns $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ with $\omega \in \mathcal{P}(\mathbf{R}^{2d})$, and are sometimes called standard modulation spaces. They were introduced by Feichtinger in [16] for certain choices of ω .

More generally, for any $\sigma \in \mathcal{S}_{2d}$, $\mathbf{p} \in (0, \infty]^{2d}$ and $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$, the modulation space $M_{\sigma,(\omega)}^{\mathbf{p}}(\mathbf{R}^d)$ is the set of all $f \in \mathcal{S}'_{1/2}(\mathbf{R}^d)$ such that $V_{\phi}f \in L_{\sigma,(\omega)}^{\mathbf{p}}(\mathbf{R}^{2d})$, and we equip $M_{\sigma,(\omega)}^{\mathbf{p}}(\mathbf{R}^d)$ with the quasi-norm

$$\|f\|_{M_{\sigma,(\omega)}^{\mathbf{p}}} \equiv \|V_{\phi}f\|_{L_{\sigma,(\omega)}^{\mathbf{p}}}. \quad (1.7)$$

For conveniency we set $M_{(\omega)}^{\mathbf{p}} = M_{(\omega)}^{p,p}$, and if $\omega = 1$ everywhere, then set

$$M^{\mathbf{p}} = M_{\sigma,(\omega)}^{\mathbf{p}}, \quad M^{p,q} = M_{(\omega)}^{p,q}, \quad W^{p,q} = W_{(\omega)}^{p,q} \quad \text{and} \quad M^p = M_{(\omega)}^p.$$

In the following propositions we list some properties for modulation spaces, and refer to [16, 17, 21, 48] for proofs.

Proposition 1.1. *Let $\sigma \in S_{2d}$ and $\mathbf{p} \in (0, \infty]^{2d}$. Then the following is true:*

- (1) *if $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$, then $\Sigma_1(\mathbf{R}^d) \subseteq M_{\sigma,(\omega)}^{\mathbf{p}}(\mathbf{R}^d) \subseteq \Sigma'_1(\mathbf{R}^d)$;*
- (2) *if $\omega \in \mathcal{P}_E(\mathbf{R}^{2d})$ satisfies (1.3) for every $r > 0$, then $\mathcal{S}_1(\mathbf{R}^d) \subseteq M_{\sigma,(\omega)}^{\mathbf{p}}(\mathbf{R}^d) \subseteq \mathcal{S}'_1(\mathbf{R}^d)$;*
- (3) *if $\omega \in \mathcal{P}(\mathbf{R}^{2d})$, then $\mathcal{S}(\mathbf{R}^d) \subseteq M_{\sigma,(\omega)}^{\mathbf{p}}(\mathbf{R}^d) \subseteq \mathcal{S}'(\mathbf{R}^d)$.*

Proposition 1.2. *Let $\sigma \in S_{2d}$, $r \in (0, 1]$, $\mathbf{p}, \mathbf{p}_j \in (0, \infty]^{2d}$ and $\omega, \omega_j, v \in \mathcal{P}_E(\mathbf{R}^{2d})$, $j = 1, 2$, be such that $r \leq \mathbf{p}$, $\mathbf{p}_1 \leq \mathbf{p}_2$, $\omega_2 \lesssim \omega_1$, and ω is v -moderate. Then the following is true:*

- (1) *if $\phi \in M_{(v)}^r(\mathbf{R}^d) \setminus 0$, then $f \in M_{\sigma,(\omega)}^{\mathbf{p}}(\mathbf{R}^d)$, if and only if (1.7) is finite. In particular, $M_{\sigma,(\omega)}^{\mathbf{p}}(\mathbf{R}^d)$ is independent of the choice of $\phi \in M_{(v)}^r(\mathbf{R}^d) \setminus 0$. Moreover, $M_{\sigma,(\omega)}^{\mathbf{p}}(\mathbf{R}^d)$ is a quasi-Banach space under the quasi-norm in (1.7), and different choices of ϕ give rise to equivalent quasi-norms.*

If in addition $\mathbf{p} \geq 1$, then $M_{\sigma,(\omega)}^{\mathbf{p}}(\mathbf{R}^d)$ is a Banach space with norm (1.7);

- (2) $M_{\sigma,(\omega_1)}^{\mathbf{p}_1}(\mathbf{R}^d) \subseteq M_{\sigma,(\omega_2)}^{\mathbf{p}_2}(\mathbf{R}^d)$.

Next we discuss Gabor expansions, and start by recalling some notions. It follows from the analysis in Chapters 11–14 in [21] that the operators in the following definition are well-defined and continuous.

Definition 1.3. Let $\Lambda = \Lambda_{[\theta]}$ be as in (1.6), $\omega, v \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that ω is v -moderate, and let $\phi, \psi \in M_{(v)}^1(\mathbf{R}^d)$.

- (1) The *analysis operator* C_ϕ^Λ is the operator from $M_{(\omega)}^\infty(\mathbf{R}^d)$ to $\ell_{(\omega)}^\infty(\Lambda)$, given by

$$C_\phi^\Lambda f \equiv \{V_\phi f(j, \iota)\}_{j, \iota \in \Lambda};$$

- (2) The *synthesis operator* D_ψ^Λ is the operator from $\ell_{(\omega)}^\infty(\Lambda)$ to $M_{(\omega)}^\infty(\mathbf{R}^d)$, given by

$$D_\psi^\Lambda c \equiv \sum_{j, \iota \in \Lambda} c_{j, \iota} e^{i\langle \cdot, \iota \rangle} \phi(\cdot - j);$$

- (3) The *Gabor frame operator* $S_{\phi, \psi}^\Lambda$ is the operator on $M_{(\omega)}^\infty(\mathbf{R}^d)$, given by $D_\psi^\Lambda \circ C_\phi^\Lambda$, i. e.

$$S_{\phi, \psi}^\Lambda f \equiv \sum_{j, \iota \in \Lambda} V_\phi f(j, \iota) e^{i\langle \cdot, \iota \rangle} \psi(\cdot - j).$$

We usually assume that $\theta_1 = \dots = \theta_d = \varepsilon > 0$, and then we set $\Lambda_\varepsilon = \Lambda_{[\theta]}$.

The proof of the following result is omitted since the result follows from Theorem 13.1.1 and other results in [21] (see also Theorem S in [20]).

Proposition 1.4. *Let Λ be as in (1.6), $v \in \mathcal{P}_E(\mathbf{R}^{2d})$ be submultiplicative, and $\phi \in M_{(v)}^1(\mathbf{R}^d) \setminus 0$. Then the following is true:*

(1) *if*

$$\{e^{i\langle \cdot, \iota \rangle} \phi(\cdot - j)\}_{j, \iota \in \Lambda} \quad \text{and} \quad \{e^{i\langle \cdot, \iota \rangle} \psi(\cdot - j)\}_{j, \iota \in \Lambda} \quad (1.8)$$

are dual frames to each others, then $\psi \in M_{(v)}^1(\mathbf{R}^d)$;

(2) *there is a constant $\varepsilon_0 > 0$ such that the frame operator $S_{\phi, \phi}^\Lambda$ is a homeomorphism on $M_{(v)}^1(\mathbf{R}^d)$ and (1.8) are dual frames, when $\Lambda = \varepsilon \mathbf{Z}^d$, $\varepsilon \in (0, \varepsilon_0]$ and $\psi = (S_{\phi, \phi}^\Lambda)^{-1} \phi$.*

We also recall the following restatement of [51, Theorem 3.7] (see also Corollaries 12.2.5 and 12.2.6 in [21] and Theorem 3.7 in [19]). Here and in what follows we let $\Lambda^2 = \Lambda \times \Lambda$.

Proposition 1.5. *Let Λ be as in (1.6), $\mathbf{p} \in (0, \infty]^{2d}$, $r \in (0, 1]$ be such that $r \leq \min(\mathbf{p})$, $\sigma \in S_{2d}$, and let $\omega, v \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that ω is v -moderate. Also let $\phi, \psi \in M_{(v)}^r(\mathbf{R}^d)$ be such that (1.8) are dual frames to each other. Then the following is true:*

(1) *The operators $S_{\phi, \psi}^\Lambda \equiv D_\psi \circ C_\phi$ and $S_{\psi, \phi}^\Lambda \equiv D_\phi \circ C_\psi$ are both the identity map on $M_{\sigma, (\omega)}^{\mathbf{p}}(\mathbf{R}^d)$, and if $f \in M_{(\omega)}^{\mathbf{p}}(\mathbf{R}^d)$, then*

$$\begin{aligned} f &= \sum_{j, \iota \in \Lambda} (V_\phi f)(j, \iota) e^{i\langle \cdot, \iota \rangle} \psi(\cdot - j) \\ &= \sum_{j, \iota \in \Lambda} (V_\psi f)(j, \iota) e^{i\langle \cdot, \iota \rangle} \phi(\cdot - j), \end{aligned} \quad (1.9)$$

with unconditional norm-convergence in $M_{\sigma, (\omega)}^{\mathbf{p}}$ when $\max(\mathbf{p}) < \infty$, and with convergence in $M_{(\omega)}^\infty$ with respect to the weak topology otherwise;*

(2) *if $f \in M_{(1/v)}^\infty(\mathbf{R}^d)$, then*

$$\|f\|_{M_{\sigma, (\omega)}^{\mathbf{p}}} \asymp \|V_\phi f\|_{\ell_{\sigma, (\omega)}^{\mathbf{p}}(\Lambda^2)} \asymp \|V_\psi f\|_{\ell_{\sigma, (\omega)}^{\mathbf{p}}(\Lambda^2)}.$$

Let v , ϕ and Λ be as in Proposition 1.4. Then $(S_{\phi, \phi}^\Lambda)^{-1} \phi$ is called the *canonical dual window of ϕ* , with respect to Λ . We have

$$S_{\phi, \phi}^\Lambda(e^{i\langle \cdot, \iota \rangle} f(\cdot - j)) = e^{i\langle \cdot, \iota \rangle} (S_{\phi, \phi}^\Lambda f)(\cdot - j),$$

when $f \in M_{(1/v)}^\infty(\mathbf{R}^d)$ and $j, \iota \in \Lambda$. The series in (1.9) are called *Gabor expansions of f* with respect to ϕ and ψ .

Remark 1.6. There are several ways to achieve dual frames (1.8) satisfying the required properties in Proposition 1.5. In fact, let $v, v_0 \in \mathcal{P}_E(\mathbf{R}^{2d})$ be submultiplicative such that ω is v -moderate and $L_{(v_0)}^1(\mathbf{R}^{2d}) \subseteq$

$L^r(\mathbf{R}^{2d})$. Then Proposition 1.4 guarantees that for some choice of $\phi, \psi \in M_{(v_0 v)}^1(\mathbf{R}^d) \subseteq M_{(v)}^r(\mathbf{R}^d)$ and lattice Λ in (1.6), the sets in (1.8) are dual frames to each others, and that $\psi = (S_{\phi, \phi}^\Lambda)^{-1}\phi$.

In the sequel we usually assume that $\Lambda = \Lambda_\varepsilon$, with $\varepsilon > 0$ small enough such that the hypotheses in Propositions 1.4 and 1.5 are fulfilled, and that the window functions and their duals belong to $M_{(v)}^r$ for every $r > 0$. This is always possible, in view of Remark 1.6.

1.5. Classes of matrices. In what follows we let Λ be a in (1.6), A be the complex matrix $(a(j, k))_{j, k \in \Lambda}$, $p, q \in (0, \infty]$, ω be a map from Λ^2 to \mathbf{R}_+ , and

$$h_{A, p, \omega}(k) \equiv \|H_{A, \omega}(\cdot, k)\|_{\ell^p},$$

$$\text{where } H_{A, \omega}(j, k) = a(j, j - k)\omega(j, j - k). \quad (1.10)$$

Definition 1.7. Let $0 < p, q \leq \infty$, Λ be as in (1.6) and let ω be a map from Λ^2 to \mathbf{R}_+ .

- (1) The set $\mathbb{U}_0(\Lambda)$ consists of matrices $(a(j, k))_{j, k \in \Lambda}$ such that at most finite numbers of $a(j, k)$ are non-zero;
- (2) The set $\mathbb{U}^{p, q}(\omega, \Lambda)$ consists of all matrices $A = (a(j, k))_{j, k \in \Lambda}$ such that

$$\|A\|_{\mathbb{U}^{p, q}(\omega, \Lambda)} \equiv \|h_{A, p, \omega}\|_{\ell^q(\Lambda)},$$

is finite, where $h_{A, p, \omega}$ is given by (1.10). Furthermore, $\mathbb{U}_0^{p, q}(\omega, \Lambda)$ is the completion of $\mathbb{U}_0(\Lambda)$ under the quasi-norm $\|\cdot\|_{\mathbb{U}^{p, q}(\omega, \Lambda)}$.

For conveniency we set $\mathbb{U}^p(\omega, \Lambda) = \mathbb{U}^{p, p}(\omega, \Lambda)$, and if $\omega = 1$ everywhere, then we set $\mathbb{U}^{p, q}(\Lambda) = \mathbb{U}^{p, q}(\omega, \Lambda)$ and $\mathbb{U}^p(\Lambda) = \mathbb{U}^p(\omega, \Lambda)$.

1.6. Pseudo-differential operators. Next we recall some properties in pseudo-differential calculus. Let $s \geq 1/2$, $a \in \mathcal{S}_s(\mathbf{R}^{2d})$, and $t \in \mathbf{R}$ be fixed. Then the pseudo-differential operator $\text{Op}_t(a)$ is the linear and continuous operator on $\mathcal{S}_s(\mathbf{R}^d)$, given by

$$(\text{Op}_t(a)f)(x) = (2\pi)^{-d} \iint a((1-t)x + ty, \xi) f(y) e^{i\langle x-y, \xi \rangle} dy d\xi. \quad (1.11)$$

For general $a \in \mathcal{S}'_s(\mathbf{R}^{2d})$, the pseudo-differential operator $\text{Op}_t(a)$ is defined as the continuous operator from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}'_s(\mathbf{R}^d)$ with distribution kernel

$$K_{a, t}(x, y) = (2\pi)^{-d/2} (\mathcal{F}_2^{-1} a)((1-t)x + ty, x - y). \quad (1.12)$$

Here $\mathcal{F}_2 F$ is the partial Fourier transform of $F(x, y) \in \mathcal{S}'_s(\mathbf{R}^{2d})$ with respect to the y variable. This definition makes sense, since the mappings

$$\mathcal{F}_2 \quad \text{and} \quad F(x, y) \mapsto F((1-t)x + ty, y - x) \quad (1.13)$$

are homeomorphisms on $\mathcal{S}'_s(\mathbf{R}^{2d})$. In particular, the map $a \mapsto K_{a, t}$ is a homeomorphism on $\mathcal{S}'_s(\mathbf{R}^{2d})$.

The standard (Kohn-Nirenberg) representation, $a(x, D) = \text{Op}(a)$, and the Weyl quantization $\text{Op}^w(a)$ of a are obtained by choosing $t = 0$ and $t = 1/2$, respectively, in (1.11) and (1.12).

Remark 1.8. By Fourier's inversion formula, (1.12) and the kernel theorem [34, Theorem 2.2] for operators from Gelfand-Shilov spaces to their duals, it follows that the map $a \mapsto \text{Op}_t(a)$ is bijective from $\mathcal{S}'_s(\mathbf{R}^{2d})$ to the set of all linear and continuous operators from $\mathcal{S}_s(\mathbf{R}^d)$ to $\mathcal{S}'_s(\mathbf{R}^{2d})$.

By Remark 1.8, it follows that for every $a_1 \in \mathcal{S}'_s(\mathbf{R}^{2d})$ and $t_1, t_2 \in \mathbf{R}$, there is a unique $a_2 \in \mathcal{S}'_s(\mathbf{R}^{2d})$ such that $\text{Op}_{t_1}(a_1) = \text{Op}_{t_2}(a_2)$. By Section 18.5 in [31], the relation between a_1 and a_2 is given by

$$\text{Op}_{t_1}(a_1) = \text{Op}_{t_2}(a_2) \iff a_2(x, \xi) = e^{i(t_1 - t_2)\langle D_x, D_\xi \rangle} a_1(x, \xi). \quad (1.14)$$

We also recall that $\text{Op}_t(a)$ is a rank-one operator, i. e.

$$\text{Op}_t(a)f = (2\pi)^{-d/2}(f, f_2)f_1, \quad f \in \mathcal{S}_s(\mathbf{R}^d), \quad (1.15)$$

for some $f_1, f_2 \in \mathcal{S}'_s(\mathbf{R}^d)$, if and only if a is equal to the t -Wigner distribution

$$W_{f_1, f_2}^t(x, \xi) \equiv \mathcal{F}(f_1(x + t \cdot) \overline{f_2(x - (1 - t) \cdot)})(\xi), \quad (1.16)$$

of f_1 and f_2 . If in addition $f_1, f_2 \in L^2(\mathbf{R}^d)$, then W_{f_1, f_2}^t takes the form

$$W_{f_1, f_2}^t(x, \xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f_1(x + ty) \overline{f_2(x - (1 - t)y)} e^{-\langle y, \xi \rangle} dy. \quad (1.17)$$

(Cf. [5].) Since the Weyl case is of peculiar interests, we also set $W_{f_1, f_2} = W_{f_1, f_2}^t$, when $t = 1/2$.

1.7. Schatten-von Neumann classes. Let $\mathcal{B}(V_1, V_2)$ denote the set of all linear and continuous operators from the quasi-normed space V_1 to the quasi-normed space V_2 , and let $\|\cdot\|_{\mathcal{B}(V_1, V_2)}$ denote corresponding quasi-norm. Let \mathcal{H}_k , $k = 1, 2, 3$, be Hilbert spaces and $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then the *singular value* of T of order $j \geq 1$ is defined by

$$\sigma_j(T) = \sigma_j(T, \mathcal{H}_1, \mathcal{H}_2) \equiv \inf \|T - T_0\|_{\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)},$$

where the infimum is taken over all linear operators T_0 from \mathcal{H}_1 to \mathcal{H}_2 of rank at most $j - 1$. The set $\mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)$ of Schatten-von Neumann operators from \mathcal{H}_1 to \mathcal{H}_2 of order $p \in (0, \infty]$ is the set of all $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that

$$\|T\|_{\mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)} \equiv \|\{\sigma_j(T)\}_{j \geq 1}\|_{\ell^p} \quad (1.18)$$

is finite. We observe that $\mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)$ is contained in the set of compact operators from \mathcal{H}_1 to \mathcal{H}_2 , when $p < \infty$.

We recall that if $p_0, p_1, p_2 \in (0, \infty]$, then

$$\begin{aligned} \|T_2 \circ T_1\|_{\mathcal{I}_{p_0}(\mathcal{H}_1, \mathcal{H}_3)} &\leq \|T_1\|_{\mathcal{I}_{p_1}(\mathcal{H}_1, \mathcal{H}_2)} \|T_2\|_{\mathcal{I}_{p_2}(\mathcal{H}_2, \mathcal{H}_3)} \quad \text{when} \\ T_1 \in \mathcal{I}_{p_1}(\mathcal{H}_1, \mathcal{H}_2), \quad T_2 \in \mathcal{I}_{p_2}(\mathcal{H}_2, \mathcal{H}_3), \quad \frac{1}{p_1} + \frac{1}{p_2} &= \frac{1}{p_0}, \end{aligned} \quad (1.19)$$

and refer to [4, 41] for more facts about Schatten-von Neumann classes.

For convenience we set

$$\mathcal{I}_p(\omega_1, \omega_2) \equiv \mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2),$$

when $\mathcal{H}_k = M_{(\omega_k)}^2(\mathbf{R}^d)$, for some $\omega_k \in \mathcal{P}_E(\mathbf{R}^{2d})$, $k = 1, 2$. Moreover, if $t \in \mathbf{R}$ and then $s_{t,p}(\omega_1, \omega_2)$ is the set of all $a \in \mathcal{S}'_{1/2}(\mathbf{R}^{2d})$ such that $\text{Op}_t(a) \in \mathcal{I}_p(\omega_1, \omega_2)$, and we set

$$\|a\|_{s_{t,p}(\omega_1, \omega_2)} \equiv \|\text{Op}_t(a)\|_{\mathcal{I}_p(\omega_1, \omega_2)}.$$

We also set $s_p^w(\omega_1, \omega_2) = s_{t,p}(\omega_1, \omega_2)$ in the Weyl case, i. e. when $t = 1/2$. Moreover, if $\omega_1 = \omega_2 = 1$, then we set $s_{t,p}(\mathbf{R}^{2d}) = s_{t,p}(\omega_1, \omega_2)$ and $s_p^w(\mathbf{R}^{2d}) = s_p^w(\omega_1, \omega_2)$.

We recall that $s_{t,p}(\omega_1, \omega_2)$ is a quasi-Banach space under the quasi-norm $a \mapsto \|a\|_{s_{t,p}(\omega_1, \omega_2)} \equiv \|\text{Op}_t(a)\|_{\mathcal{I}_p(\omega_1, \omega_2)}$. Furthermore, if in addition $p \geq 1$, then $s_{t,p}(\omega_1, \omega_2)$ is a Banach space.

By Remark 1.8 it follows that the map $a \mapsto \text{Op}_t(a)$ from $s_{t,p}(\omega_1, \omega_2)$ to $\mathcal{I}_p(\omega_1, \omega_2)$ is bijective and norm preserving.

1.8. Symplectic vector spaces and Hörmander symbol classes.

A real vector space W of dimension $2d$ is called *symplectic* if there is a non-degenerate and anti-symmetric bilinear form σ (the symplectic form). By choosing symplectic coordinates $e_1, \dots, e_d, \varepsilon_1, \dots, \varepsilon_d$ in W , it follows that

$$\sigma(X, Y) = \langle y, \xi \rangle - \langle x, \eta \rangle,$$

with

$$X = (x, \xi) = \sum_{k=1}^d (x_k e_k + \xi_k \varepsilon_k) \in W, \quad Y = (y, \eta) = \sum_{k=1}^d (y_k e_k + \eta_k \varepsilon_k) \in W,$$

which allows us to identify W with the phase space T^*V for some vector space V of dimension d , or by $T^*\mathbf{R}^d \simeq \mathbf{R}^{2d}$.

The symplectic Fourier transform \mathcal{F}_σ is the linear and continuous map on $\mathcal{S}'_{1/2}(W)$, given by

$$(\mathcal{F}_\sigma a)(X) = \pi^{-d} \int_W a(Y) e^{2i\sigma(X, Y)} dY$$

when $a \in \mathcal{S}(W)$. If $X = (x, \xi) \in T^*\mathbf{R}^d = W$, then it follows that $(\mathcal{F}_\sigma a)(X) = 2^d \widehat{a}(-2\xi, 2x)$.

Next we recall some notions on Hörmander symbol classes, $S(m, g)$, parameterized by the Riemannian metric g and the weight function m on the $2d$ dimensional symplectic vector space W (see e. g. [7, 8, 30, 31, 33, 47]). The reader who is not interested of the Schatten-von Neumann results in Section 4 of pseudo-differential operators with symbols in $S(m, g)$ may pass to the next section.

The Hörmander class $S(m, g)$ consists of all $a \in C^\infty(W)$ such that

$$\|a\|_{m,N}^g \equiv \sum_{k=0}^N \sup_{X \in W} (|a|_k^g(X)/m(X)), \quad \text{where} \quad |a|_k^g(X) = \sup |a^{(k)}(X; Y_1, \dots, Y_k)|.$$

Here the latter supremum is taken over all $Y_1, \dots, Y_k \in W$ such that $g_X(Y_j) \leq 1$, $j = 1, \dots, k$, and $|a|_0^g(X)$ is interpreted as $|a(X)|$.

We need to add some conditions on m and g . The metric g is called *slowly varying* if there are positive constants c and C such that

$$C^{-1}g_X \leq g_Y \leq Cg_X, \quad \text{when} \quad X, Y \in W \quad (1.20)$$

satisfy $g_X(X - Y) \leq c$, and m is called g -continuous when (1.20) holds with $m(X)$ and $m(Y)$ in place of g_X and g_Y , respectively, provided $g_X(X - Y) \leq c$.

For the Riemannian metric g on W , the *dual* metric g^σ with respect to the symplectic form σ , and the *Planck's function* h_g are defined by

$$g_X^\sigma(Z) \equiv \sup_{g_X(Y) \leq 1} \sigma(Y, Z)^2 \quad \text{and} \quad h_g(X) \equiv \sup_{g_X^\sigma(Y) \leq 1} g_X(Y)^{1/2}.$$

Moreover, if g is slowly varying and m is g -continuous, then g is called σ -temperate if there are positive constants C and N such that

$$g_Y(Z) \leq Cg_X(Z)(1 + g_Y(X - Y))^N, \quad X, Y, Z \in W, \quad (1.21)$$

and m is called (σ, g) -temperate if it is g -continuous and (1.21) holds with $m(X)$ and $m(Y)$ in place of $g_X(Z)$ and $g_Y(Z)$, respectively.

Definition 1.9. Let g be a Riemannian metric on W . Then g is called *feasible* if it is slowly varying and $h_g \leq 1$ everywhere. Furthermore, g is called *strongly feasible* if it is feasible and σ -temperate.

We remark that the Hörmander class $S_{\rho, \delta}^r$ in [31], the SG-class in [11, 36], the Shubin classes in [40, Definition 23.1] and other well-known families of symbol classes are given by $S(m, g)$ for suitable choices of strongly feasible metrics g and (σ, g) -temperate weights m .

2. ESTIMATES FOR MATRICES

In this section we deduce continuity and Schatten-properties for matrices in the classes $\mathbb{U}^{p,q}(\omega, \Lambda)$. In the first part we achieve convenient factorization results for matrices in the case when $p = q$ (cf. Theorem 2.1). Thereafter we establish the continuity properties (cf. Theorem 2.3). In the last part of the section we combine these factorizations and continuity results to establish Schatten properties for matrix operators (cf. Theorem 2.5).

Theorem 2.1 below allows factorizations of matrices in $\mathbb{U}^p(\omega, \Lambda)$ in suitable ways, when Λ is given by (1.6). Here the involved weights should fulfill

$$\omega_1(j, j)\omega_2(j, k) \leq \omega_0(j, k), \quad j, k \in \Lambda \quad (2.1)$$

or

$$\omega_1(j, k)\omega_2(k, k) \leq \omega_0(j, k), \quad j, k \in \Lambda \quad (2.2)$$

and the involved Lebesgue exponents should satisfy the Hölder condition

$$\frac{1}{p_0} \leq \frac{1}{p_1} + \frac{1}{p_2}, \quad (2.3)$$

Theorem 2.1. *Let Λ be as in (1.6), $p_l \in (0, \infty]$ be such that (2.3) hold, ω_l , $l = 0, 1, 2$, be weights on \mathbf{R}^{2d} , and let $A_0 \in \mathbb{U}^{p_0}(\omega_0, \Lambda)$. Then the following is true:*

- (1) *if (2.1) holds, then $A_0 = A_1 \cdot A_2$ for some $A_l \in \mathbb{U}^{p_l}(\omega_l, \Lambda)$, $l = 1, 2$. Furthermore, A_1 can be chosen as a diagonal matrix;*
- (2) *if (2.2) holds, then $A_0 = A_1 \cdot A_2$ for some $A_l \in \mathbb{U}^{p_l}(\omega_l, \Lambda)$, $l = 1, 2$. Furthermore, A_2 can be chosen as a diagonal matrix.*

Moreover, the matrices in (1) and (2) can be chosen such that

$$\|A_1\|_{\mathbb{U}^{p_1}(\omega_1, \Lambda)} \|A_2\|_{\mathbb{U}^{p_2}(\omega_2, \Lambda)} \leq \|A_0\|_{\mathbb{U}^{p_0}(\omega_0, \Lambda)}. \quad (2.4)$$

Proof. It is no restrictions to assume that equality is attained in (2.3), and by transposition it also suffices to prove (1).

We only prove the result for $p_0 < \infty$. The small modifications to the case when $p_0 = \infty$ are left for the reader. Let $a(j, k)$ be the matrix elements for A_0 , and let $A_1 = (b(j, k))$ and $A_2 = (c(j, k))$ be the matrices such that

$$b(j, k) = \begin{cases} (\omega_1(j, j))^{-1} \left(\sum_m |a(j, m)\omega_0(j, m)|^{p_0} \right)^{1/p_1}, & j = k \\ 0, & j \neq k \end{cases}$$

and $c(j, k) = a(j, k)/b(j, j)$ when $b(j, j) \neq 0$, and $c(j, k) = 0$ otherwise. Since

$$b(j, j) \geq (\omega_1(j, j))^{-1} |a(j, j)\omega_0(j, j)|^{p_0/p_1}, \quad \text{and} \quad \frac{1}{p_0} - \frac{1}{p_1} = \frac{1}{p_2},$$

(2.1) gives

$$\begin{aligned} |c(j, k)\omega_2(j, k)| &\leq |a(j, k)|^{p_0/p_2} \omega_1(j, j)\omega_2(j, k)/\omega_0(j, k)^{p_0/p_1} \\ &\leq |a(j, k)|^{p_0/p_2} \omega_0(j, k)^{p_0/p_2}. \end{aligned}$$

This in turn gives

$$\begin{aligned}\|A_1\|_{\mathbb{U}^{p_1}(\omega_1, \Lambda)} &= \left(\sum_{j,k} |b(j,k)\omega_1(j,k)|^{p_1} \right)^{1/p_1} \\ &= \left(\left(\sum_j \left(\sum_m |a(j,m)\omega_0(j,m)|^{p_0} \right)^{1/p_1} \right)^{p_1} \right)^{1/p_1} = \|A_0\|_{\mathbb{U}^{p_0}(\omega_0, \Lambda)}^{p_0/p_1},\end{aligned}$$

and

$$\begin{aligned}\|A_2\|_{\mathbb{U}^{p_2}(\omega_2, \Lambda)} &= \left(\sum_{j,k} |c(j,k)\omega_2(j,k)|^{p_2} \right)^{1/p_2} \\ &\leq \left(\sum_{j,k} |a(j,k)\omega_0(j,k)|^{p_0} \right)^{1/p_2} = \|A_0\|_{\mathbb{U}^{p_0}(\omega_0, \Lambda)}^{p_0/p_2}.\end{aligned}$$

Hence $A_l \in \mathbb{U}^{p_l}(\omega_l, \Lambda)$, $l = 1, 2$. Since $A_0 = A_1 \cdot A_2$ and $p_0/p_1 + p_0/p_2 = 1$, the result follows. \square

If the weights ω_l , $l = 0, 1, 2$, fulfill

$$\omega_1(j,m)\omega_2(m,k) \leq \omega_0(j,k), \quad \text{for every } j, k, m \in \Lambda, \quad (2.5)$$

then it is evident that both (2.1) and (2.2) are fulfilled. Hence the following result is a special case of Theorem 2.1.

Proposition 2.2. *Let Λ be as in (1.6), $p_l \in (0, \infty]$ and let ω_l , $l = 0, 1, 2$, be weights on \mathbf{R}^{2d} such that (2.3) and (2.5) hold, and let $A_0 \in \mathbb{U}^{p_0}(\omega_0, \Lambda)$. Then $A_0 = A_1 \cdot A_2$ for some $A_l \in \mathbb{U}^{p_l}(\omega_l, \Lambda)$, $l = 1, 2$. Moreover, the matrices A_1 and A_2 can be chosen such that (2.4) holds.*

Next we deduce continuity results for matrix operators. We recall that if $A = (a(j,k))_{j,k \in \Lambda}$ is a matrix, then Af is uniquely defined as an element in $\ell(\Lambda)$ when $f \in \ell_0(\Lambda)$, i. e.

$$A : \ell_0(\Lambda) \mapsto \ell(\Lambda). \quad (2.6)$$

Furthermore, if in addition A belongs to $\mathbb{U}_0(\Lambda)$, then Af is uniquely defined as an element in $\ell_0(\Lambda)$ when $f \in \ell(\Lambda)$, i. e.

$$A : \ell(\Lambda) \mapsto \ell_0(\Lambda) \quad \text{when } A \in \mathbb{U}_0(\Lambda). \quad (2.7)$$

For $p \in [1, \infty]$, its conjugate exponent $p' \in [1, \infty]$ is usually defined by $1/p + 1/p' = 1$. For p belonging to the larger interval $(0, \infty]$ it is

convenient to extend the definition of p' as

$$p' = \begin{cases} 1, & p = \infty \\ \frac{p}{p-1}, & 1 < p < \infty \\ \infty, & 0 < p \leq 1. \end{cases}$$

The next theorem is the main result concerning the continuity for matrix operators.

Theorem 2.3. *Let $\sigma \in S_d$, $\theta \in \mathbf{R}_*^d$, $\Lambda = T_\theta \mathbf{Z}^d$, ω_l be weights on Λ , $l = 1, 2$, and ω_0 be a weight on $\Lambda \times \Lambda$ such that (2.14) holds. Also let $\mathbf{p}_1, \mathbf{p}_2 \in (0, \infty]^n$, and $p, q \in (0, \infty]$ be such that*

$$\frac{1}{\mathbf{p}_2} - \frac{1}{\mathbf{p}_1} = \frac{1}{p} + \min\left(0, \frac{1}{q} - 1\right), \quad q \leq \min(\mathbf{p}_2) \leq \max(\mathbf{p}_2) \leq p, \quad (2.8)$$

and let $A \in \mathbb{U}^{p,q}(\omega_0, \Lambda)$. Then A from $\ell_0(\Lambda)$ to $\ell(\Lambda)$ is uniquely extendable to a continuous map from $\ell_{\sigma,(\omega_1)}^{\mathbf{p}_1}(\Lambda)$ to $\ell_{\sigma,(\omega_2)}^{\mathbf{p}_2}(\Lambda)$, and

$$\|A\|_{\mathcal{B}(\ell_{\sigma,(\omega_1)}^{\mathbf{p}_1}(\Lambda), \ell_{\sigma,(\omega_2)}^{\mathbf{p}_2}(\Lambda))} \leq \|A\|_{\mathbb{U}^{p,q}(\omega_0, \Lambda)}. \quad (2.9)$$

We note that (2.9) is the same as

$$\|Af\|_{\ell_{\sigma,(\omega_2)}^{\mathbf{p}_2}(\Lambda)} \leq \|A\|_{\mathbb{U}^{p,q}(\omega_0, \Lambda)} \|f\|_{\ell_{\sigma,(\omega_1)}^{\mathbf{p}_1}(\Lambda)}, \quad f \in \ell_{\sigma,(\omega_1)}^{\mathbf{p}_1}(\Lambda). \quad (2.10)$$

Proof. By permutation of the Lebesgue exponents, we reduce ourself to the case when σ is the identity map. We consider the cases $q \leq 1$ and $q \geq 1$ separately. Let $f \in \ell_{(\omega_1)}^{\mathbf{p}_1}$, $h = h_{A,\infty,\omega}$ be the same as in Definition 1.7, with $\omega = \omega_0$, and set

$$c(k) = |f(k)\omega_1(k)|, \quad a_0(j, k) = |a(j, j-k)\omega_0(j, j-k)| \quad \text{and} \quad g = Af.$$

First we consider the case when $p = \infty$, $q \leq 1$, and in addition $A \in \mathbb{U}_0(\Lambda)$. Then $\mathbf{p}_1 = \mathbf{p}_2$, and we get

$$\begin{aligned} |g(j)\omega_2(j)| &\leq \sum_k |a(j, k)\omega_0(j, k)| c(k) \\ &= \sum_k a_0(j, k) c(j-k) \\ &\leq \sum_k h(k) c(j-k) = (h * c)(j). \end{aligned} \quad (2.11)$$

Hence, Corollary 2.2 in [51] gives

$$\|Af\|_{\ell_{(\omega_2)}^{\mathbf{p}_2}} = \|g \cdot \omega_2\|_{\ell^{\mathbf{p}_2}} \leq \|h\|_{\ell^q} \|c\|_{\ell^{\mathbf{p}_1}} = \|A\|_{\mathbb{U}^{\infty,q}(\omega_0, \Lambda)} \|f\|_{\ell_{(\omega_1)}^{\mathbf{p}_1}}, \quad (2.12)$$

and the result follows in this case.

For general $A \in \mathbb{U}^{\infty,q}(\omega_0, \Lambda)$ we decompose A and f into

$$A = A_1 - A_2 + i(A_3 - A_4) \quad \text{and} \quad f = f_1 - f_2 + i(f_3 - f_4),$$

where A_j and f_k only have non-negative entries, chosen as small as possible. By Beppo Levi's theorem and the estimates above it follows that $A_j f_k$ is uniquely defined as an element in $\ell_{(\omega_1)}^{p_1}$. It also follows from these estimates (2.9) holds, and we have proved the result in the case $p = \infty$ and $q \leq 1$.

The case when $q \leq 1, p < \infty$ and $A \in \mathbb{U}_0(\Lambda)$ is obtained by induction. Let

$$\begin{aligned} G_0(j) &\equiv |g(j)\omega_2(j)|, \quad b_0(j, k) \equiv a_0(j, k) = |a(j, j-k)\omega_0(j, j-k)|, \\ c_0(j) &\equiv c(j), \quad \Lambda_0 \equiv \{0\} \quad \text{and} \quad \Lambda_0^* \equiv \Lambda = \theta_1 \mathbf{Z} \times \cdots \times \theta_d \mathbf{Z}. \end{aligned}$$

Also let

$$\begin{aligned} \mathbf{p}_{l,m} &\equiv (p_{l,1}, \dots, p_{l,m}) \quad \text{when} \quad \mathbf{p}_l = (p_{l,1}, \dots, p_{l,d}), \quad l = 1, 2, \\ \Lambda_m &\equiv \theta_1 \mathbf{Z} \times \cdots \times \theta_m \mathbf{Z}, \quad \Lambda_m^* \equiv \theta_{m+1} \mathbf{Z} \times \cdots \times \theta_d \mathbf{Z}, \\ \mathbf{j}_m &= (j_{m+1}, \dots, j_d) \in \Lambda_m^* \quad \text{and} \quad \mathbf{k}_m = (k_{m+1}, \dots, k_d) \in \Lambda_m^* \end{aligned}$$

when

$$j = (j_1, \dots, j_d) \in \Lambda \quad \text{and} \quad k = (k_1, \dots, k_d) \in \Lambda,$$

and let

$$b_m(\mathbf{j}_m, \mathbf{k}_m) \equiv \|a_{0,m}(\mathbf{j}_m, \cdot, \mathbf{k}_m)\|_{\ell^q(\Lambda_m)}, \quad m = 1, \dots, d,$$

where

$$a_{0,m}(\mathbf{j}_m, k) \equiv \|a_0(\cdot, \mathbf{j}_m, k)\|_{\ell^p(\Lambda_m)}.$$

Define inductively

$$G_m(\mathbf{j}_m) \equiv \|G_{m-1}(\cdot, \mathbf{j}_m)\|_{\ell^{p_2,m}(\theta_m \mathbf{Z})} \quad \text{and} \quad c_m(\mathbf{j}_m) \equiv \|c_{m-1}(\cdot, \mathbf{j}_m)\|_{\ell^{p_1,m}(\theta_m \mathbf{Z})},$$

when $m = 1, \dots, d$, where Λ_d^* , G_d and c_d are interpreted as

$$\{0\}, \quad \|G_{d-1}\|_{\ell^{p_2,d}(\theta_d \mathbf{Z})} = \|Af\|_{\ell_{(\omega_2)}^{p_2}} \quad \text{and} \quad \|c_{d-1}\|_{\ell^{p_1,d}(\theta_d \mathbf{Z})} = \|f\|_{\ell_{(\omega_1)}^{p_1}},$$

respectively. We claim

$$G_m(\mathbf{j}_m) \leq \left(\sum_{\mathbf{k}_m \in \Lambda_m^*} (b_m(\mathbf{j}_m, \mathbf{k}_m) c_m(\mathbf{j}_m - \mathbf{k}_m))^q \right)^{1/q} \quad (2.13)$$

for $m = 0, \dots, d$.

In fact, the case $m = 0$ follows from the equality in (2.11) and the fact that $q \leq 1$. Suppose (2.13) is true for $m - 1$ in place of m , and let $r = p_{2,m}/q$. Then $r \in [1, \infty)$, since $p < \infty$ and $q \leq p_{2,m}$. Hence, (2.8), and Hölder's and Minkowski's inequalities in combination with

the inductive assumptions give

$$\begin{aligned}
G_m(\mathbf{j}_m)^q &\leq \left(\sum_{\mathbf{j}_m \in \theta_m \mathbf{Z}} \left(\sum_{\mathbf{k}_{m-1} \in \Lambda_{m-1}^*} (b_{m-1}(\mathbf{j}_{m-1}, \mathbf{k}_{m-1}) c_{m-1}(\mathbf{j}_{m-1} - \mathbf{k}_{m-1}))^q \right)^r \right)^{1/r} \\
&\leq \sum_{\mathbf{k}_{m-1} \in \Lambda_{m-1}^*} \left(\sum_{\mathbf{j}_m \in \theta_m \mathbf{Z}} (b_{m-1}(\mathbf{j}_{m-1}, \mathbf{k}_{m-1}) c_{m-1}(\mathbf{j}_{m-1} - \mathbf{k}_{m-1}))^{p_2, m} \right)^{1/r} \\
&\leq \sum_{\mathbf{k}_{m-1} \in \Lambda_{m-1}^*} \|b_{m-1}(\cdot, \mathbf{j}_m, \mathbf{k}_{m-1})\|_{\ell^p(\theta_m \mathbf{Z})}^q \|c_{m-1}(\cdot, \mathbf{j}_m - \mathbf{k}_{m-1})\|_{\ell^{p_1, m}(\theta_m \mathbf{Z})}^q.
\end{aligned}$$

We have $c_m(\mathbf{j}_m) = \|c_{m-1}(\cdot, \mathbf{j}_m)\|_{\ell^{p_1, m}(\theta_m \mathbf{Z})}$, and

$$\sum_{\mathbf{k}_m \in \theta_m \mathbf{Z}} \|b_{m-1}(\cdot, \mathbf{j}_m, \mathbf{k}_m, \mathbf{k}_m)\|_{\ell^p(\theta_m \mathbf{Z})}^q \leq \|a_{0, m}(\mathbf{j}_m \cdot, \mathbf{k}_m)\|_{\ell^p(\theta_m \mathbf{Z})}^q,$$

by Minkowski's inequality, and a combination of the previous inequalities give (2.13). Hence, by induction we have that (2.13) holds for every $m = 0, \dots, d$, and by letting $m = d$ we obtain (2.9) when $A \in \mathbb{U}_0(\Lambda)$. The result now follows for general $\mathbb{U}^{p, q}(\omega_0, \Lambda)$ when $p < \infty$ and $q \leq 1$ by the fact that $\mathbb{U}_0(\Lambda)$ is dense in $\mathbb{U}^{p, q}(\omega_0, \Lambda)$.

Next we consider the case $q \in (1, \infty]$, and assume first that $p = \infty$. Then

$$\frac{1}{\mathbf{p}_1} + \frac{1}{q} = 1 + \frac{1}{\mathbf{p}_2}.$$

Hence, if $A \in \mathbb{U}^{\infty, q}(\omega_0, \Lambda)$ and $f \in \ell_0(\Lambda)$, then (2.11) and Young's inequality give

$$\|g\|_{\ell_{(\omega_2)}^{p_2}} \leq \|h\|_{\ell^q} \|c\|_{\ell^{\mathbf{p}_1}},$$

and (2.9) follows in this case as well. Since $\max(\mathbf{p}_1) < \infty$ when $q > 1$, the result follows for general $f \in \ell_{(\omega_1)}^{\mathbf{p}_1}(\Lambda)$ from the fact that $\ell_0(\Lambda)$ is dense in $\ell_{(\omega_1)}^{\mathbf{p}_1}(\Lambda)$.

For general p , the result now follows by multi-linear interpolation between the cases $(p, q) = (1, 1)$ and $(p, q) = \{\infty\} \times [1, \infty]$, using Theorems 4.4.1 and 5.6.3 in [3]. The proof is complete. \square

The following consequence of the previous result is particularly important.

Corollary 2.4. *Let Λ be as in (1.6), $p \in (0, \infty]$, ω_l , $l = 1, 2$ be weights on \mathbf{R}^d and ω_0 be a weight on \mathbf{R}^{2d} such that*

$$\frac{\omega_2(j)}{\omega_1(k)} \leq \omega_0(j, k), \quad j, k \in \Lambda \tag{2.14}$$

holds. Also let $A \in \mathbb{U}^p(\omega_0, \Lambda)$. Then A in (2.6) is uniquely extendable to a continuous map from $\ell_{(\omega_1)}^{p'}(\Lambda)$ to $\ell_{(\omega_2)}^p(\Lambda)$, and

$$\|A\|_{\mathcal{B}(\ell_{(\omega_1)}^{p'}(\Lambda), \ell_{(\omega_2)}^p(\Lambda))} \leq \|A\|_{\mathbb{U}^p(\omega_0, \Lambda)}. \quad (2.15)$$

The next result deals with Schatten-von Neumann properties for matrix operators.

Theorem 2.5. *Let Λ be as in (1.6), ω_l , $l = 1, 2$ be weights on \mathbf{R}^d and ω_0 be a weight on \mathbf{R}^{2d} such that (2.14) holds. Also let $p \in (0, 2]$, and let $A \in \mathbb{U}^p(\omega_0, \Lambda)$. Then $A \in \mathcal{S}_p(\ell_{(\omega_1)}^2(\Lambda), \ell_{(\omega_2)}^2(\Lambda))$, and*

$$\|A\|_{\mathcal{S}_p(\ell_{(\omega_1)}^2(\Lambda), \ell_{(\omega_2)}^2(\Lambda))} \leq \|A\|_{\mathbb{U}^p(\omega_0, \Lambda)}. \quad (2.16)$$

Proof. We may assume that equality is attained in (2.14), and that $\|A\|_{\mathbb{U}^p(\omega_0, \Lambda)} = 1$. Then it follows that

$$\mathcal{S}_2(\ell_{(\omega_1)}^2(\Lambda), \ell_{(\omega_2)}^2(\Lambda)) = \mathbb{U}^2(\omega_0, \Lambda),$$

with equality in norms.

First assume that $p = 2/N$ for some integer $N \geq 3$, and let $A \in \mathbb{U}^{2/N}(\omega_0, \Lambda)$. Also let $\vartheta_1(j, k) = \omega_2(j)$, $\vartheta_m(j, k) = 1$, $j = 2, \dots, N-1$ and $\vartheta_N(j, k) = \omega_1(k)$. By Theorem 2.1 we have

$$A = A_1 \circ \dots \circ A_N$$

for some $A_m \in \mathbb{U}^2(\vartheta_m, \Lambda)$ which satisfy $\|A_m\|_{\mathbb{U}^2(\vartheta_m, \Lambda)} \leq 1$, $m = 1, \dots, N$.

By (1.19) we get

$$\begin{aligned} \|A\|_{\mathcal{S}_{2/N}(\ell_{(\omega_1)}^2, \ell_{(\omega_2)}^2)} &\leq \|A_1\|_{\mathcal{S}_2(\ell^2, \ell_{(\omega_2)}^2)} \|A_N\|_{\mathcal{S}_2(\ell_{(\omega_1)}^2, \ell^2)} \prod_{m=2}^{N-1} \|A_m\|_{\mathcal{S}_2(\ell^2, \ell^2)} \\ &= \prod_{m=2}^{N-1} \|A_m\|_{\mathbb{U}^2(\vartheta_m, \Lambda)} \leq 1, \end{aligned}$$

and the result follows in the case $p = 2/N$.

The result is therefore true when $p = 2/N$ for some integer $N \geq 3$, and when $p = 2$. For $p \in [2/N, 2]$, the result now follows by (real) interpolation between the cases $p = 2$ and $p = 2/N$, letting q , p_θ , p_k , q_k and $\theta \in [0, 1]$, $k = 0, 1$, in Teorema 3.2, (3.11) and (3.13) in [4] be chosen such that

$$q = p_\theta, \quad q_0 = p_0 = \frac{2}{N}, \quad q_1 = p_1 = 2 \quad \text{and} \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

For general $p \in (0, 2]$, the result now follows by choosing $N \geq 3$ such that $p > 2/N$. The proof is complete. \square

3. CONTINUITY AND SCHATTEN-VON NEUMANN PROPERTIES FOR PSEUDO-DIFFERENTIAL OPERATORS

In this section we deduce continuity and Schatten-von Neumann results for pseudo-differential operators with symbols in modulation spaces. In particular we extend results in [21, 25, 45, 48, 50] to include Schatten and Lebesgue parameters less than one.

We start with the following result on continuity.

Theorem 3.1. *Let $t \in \mathbf{R}$, $\sigma \in S_{2d}$, $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $\omega_0 \in \mathcal{P}_E(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ be such that*

$$\frac{\omega_2(x, \xi)}{\omega_1(y, \eta)} \lesssim \omega_0((1-t)x + ty, t\xi + (1-t)\eta, \xi - \eta, y - x).$$

Also let $\mathbf{p}_1, \mathbf{p}_2 \in (0, \infty]^{2d}$, $p, q \in (0, \infty]$ be such that (2.8) hold, and let $a \in M_{(\omega_0)}^{p,q}(\mathbf{R}^{2d})$. Then $\text{Op}_t(a)$ from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}'_{1/2}(\mathbf{R}^d)$ extends uniquely to a continuous map from $M_{\sigma,(\omega_1)}^{\mathbf{p}_1}(\mathbf{R}^d)$ to $M_{\sigma,(\omega_2)}^{\mathbf{p}_2}(\mathbf{R}^d)$, and

$$\|\text{Op}_t(a)\|_{\mathcal{B}(M_{\sigma,(\omega_1)}^{\mathbf{p}_1}, M_{\sigma,(\omega_2)}^{\mathbf{p}_2})} \lesssim \|a\|_{M_{(\omega_0)}^{p,q}}. \quad (3.1)$$

We need some preparing lemmata for the proof. We recall that $\Lambda^2 = \Lambda \times \Lambda$ when Λ is a lattice.

Lemma 3.2. *Let $v \in \mathcal{P}_E(\mathbf{R}^{4d})$, $\phi_1, \phi_2 \in \Sigma_1(\mathbf{R}^d) \setminus 0$, and let*

$$\Phi(x, \xi) = \phi_1(x) \overline{\phi_2(\xi)} e^{-i\langle x, \xi \rangle},$$

Then there is a lattice Λ in (1.6) such that

$$\{\Phi(x - j, \xi - \iota) e^{i(\langle x, \kappa \rangle + \langle k, \xi \rangle)}\}_{(j, \iota), (k, \kappa) \in \Lambda^2}$$

is a Gabor frame with canonical dual frame

$$\{\Psi(x - j, \xi - \iota) e^{i(\langle x, \kappa \rangle + \langle k, \xi \rangle)}\}_{(j, \iota), (k, \kappa) \in \Lambda^2},$$

where $\Psi = (S_{\Phi, \Phi}^{\Lambda^2 \times \Lambda^2})^{-1} \Phi$ belongs to $M_{(v)}^r(\mathbf{R}^{2d})$ for every $r > 0$.

Note that Φ in Lemma 3.2 is the Rihaczek (cross)-distribution of ϕ_1 and ϕ_2 (cf. [27]).

Proof. The result follows from Remark 1.6, and the fact that $\Phi \in \Sigma_1(\mathbf{R}^{2d}) \setminus 0$ in view of [9, Theorem 3.1] or [10, Proposition 3.4]. \square

Lemma 3.3. *Let Λ , ϕ_1 , ϕ_2 , Φ and Ψ be as in Lemma 3.2. Also let $v \in \mathcal{P}_E(\mathbf{R}^{4d})$, $a \in M_{(1/v)}^\infty(\mathbf{R}^{2d})$,*

$$c_0(\mathbf{j}, \mathbf{k}) \equiv (V_\Psi a)(j, \kappa, \iota - \kappa, k - j) e^{i\langle k - j, \kappa \rangle},$$

$$\text{where } \mathbf{j} = (j, \iota) \in \Lambda^2, \mathbf{k} = (k, \kappa) \in \Lambda^2.$$

and let A be the matrix $A = (c_0(\mathbf{j}, \mathbf{k}))_{\mathbf{j}, \mathbf{k} \in \Lambda^2}$. Then the following is true:

(1) if $p, q \in (0, \infty]$ and $\omega, \omega_0 \in \mathcal{P}_E(\mathbf{R}^{4d})$ satisfy

$$\omega(x, \xi, y, \eta) \asymp \omega_0(x, \eta, \xi - \eta, y - x), \quad (3.2)$$

then $a \in M_{(\omega_0)}^{p,q}(\mathbf{R}^{2d})$, if and only if $A \in \mathbb{U}^{p,q}(\omega, \Lambda^2)$, and then

$$\|a\|_{M_{(\omega_0)}^{p,q}} \asymp \|A\|_{\mathbb{U}^{p,q}(\omega, \Lambda^2)};$$

(2) $\text{Op}(a)$ as a map from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}'_{1/2}(\mathbf{R}^d)$, given by

$$\text{Op}(a) = D_{\phi_1} \circ A \circ C_{\phi_2}. \quad (3.3)$$

Some arguments in [27] appear in the proof of Lemma 3.3.

Proof. We have

$$|c_0(\mathbf{j}, \mathbf{j} - \mathbf{k})| = |(V_\Psi a)(j, \iota - \kappa, \kappa, -k)|.$$

Hence, Proposition 1.5 (2) gives

$$\|A\|_{\mathbb{U}^{p,q}(\omega, \Lambda^2)} = \|V_\Psi a\|_{\ell_{(\omega_0)}^{p,q}(\Lambda^2 \times \Lambda^2)} \asymp \|a\|_{M_{(\omega_0)}^{p,q}},$$

and (1) follows.

Next we prove (2). Let $f \in \mathcal{S}_{1/2}(\mathbf{R}^d)$, and let

$$c(\mathbf{j}, \mathbf{k}) = (V_\Psi a)(j, \iota, \kappa, k).$$

By Proposition 1.5 we have

$$a = \sum_{\mathbf{j}, \mathbf{k} \in \Lambda^2} c(\mathbf{j}, \mathbf{k}) \Phi_{\mathbf{j}, \mathbf{k}},$$

where

$$\Phi_{\mathbf{j}, \mathbf{k}}(x, \xi) = e^{i(\langle x, \kappa \rangle + \langle k, \xi \rangle)} \Phi(x - j, \xi - \iota).$$

This gives

$$\text{Op}(a) = \sum_{\mathbf{j}, \mathbf{k} \in \Lambda^2} c(\mathbf{j}, \mathbf{k}) \text{Op}(\Phi_{\mathbf{j}, \mathbf{k}}),$$

and we shall evaluate $\text{Op}(\Phi_{\mathbf{j}, \mathbf{k}})f$.

We have

$$\Phi_{\mathbf{j}, \mathbf{k}}(x, \xi) = \phi_1(x - j) \overline{\widehat{\phi}_2(\xi - \iota)} e^{-i\langle x - j, \xi - \iota \rangle} e^{i(\langle x, \kappa \rangle + \langle k, \xi \rangle)},$$

and by straight-forward computations we get

$$(\text{Op}(\Phi_{\mathbf{j}, \mathbf{k}})f)(x) = \phi_1(x - j) e^{i\langle x, \iota + \kappa \rangle} e^{-i\langle j, \iota \rangle} F_0(\mathbf{j}, \mathbf{k}),$$

where

$$F_0(\mathbf{j}, \mathbf{k}) = (2\pi)^{-d/2} \int \widehat{f}(\xi) \overline{\widehat{\phi}_2(\xi - \iota)} e^{i\langle j + k, \xi \rangle} d\xi = (V_{\widehat{\phi}_2} \widehat{f})(\iota, -(j + k)).$$

Since

$$(V_{\widehat{\phi}_2} \widehat{f})(\xi, -x) = e^{i\langle x, \xi \rangle} V_{\phi_2} f(x, \xi),$$

we get

$$(\text{Op}(\Phi_{\mathbf{j}, \mathbf{k}})f)(x) = (e^{i\langle k, \iota \rangle} V_{\phi_2} f(j + k, \iota)) \phi_1(x - j) e^{i\langle x, \iota + \kappa \rangle}.$$

This gives

$$\begin{aligned}
(\text{Op}(a)f)(x) &= \sum_{j, \mathbf{k} \in \Lambda^2} (V_\Psi a)(j, \iota, \kappa, k) e^{i\langle k, \iota \rangle} V_{\phi_2} f(j+k, \iota) \phi_1(x-j) e^{i\langle x, \iota + \kappa \rangle} \\
&= \sum_{j, \mathbf{k} \in \Lambda^2} (V_\Psi a)(j, \kappa, \iota - \kappa, k-j) e^{i\langle k-j, \kappa \rangle} V_{\phi_2} f(\mathbf{k}) \phi_1(x-j) e^{i\langle x, \iota \rangle} \\
&= \sum_{j \in \Lambda^2} h(j) \phi_1(x-j) e^{i\langle x, \iota \rangle}, \quad (3.4)
\end{aligned}$$

where

$$h(j) = \sum_{\mathbf{k} \in \Lambda^2} c_0(j, \mathbf{k}) V_{\phi_2} f(\mathbf{k}).$$

The result now follows from the facts that $h = A \cdot (C_{\phi_2} f)$ and that the right-hand side of (3.4) is equal to $(D_{\phi_1} h)(x)$. \square

Proof of Theorem 3.1. By Proposition 1.7 in [51] and its proof, it suffices to prove the result for $t = 0$. Let $\omega, \omega_0, \Lambda, \phi_1, \phi_2$ and A be as in Lemma 3.3. Then

$$C_{\phi_2} : M_{(\omega_1)}^{\mathbf{p}_1}(\mathbf{R}^d) \rightarrow \ell_{(\omega_1)}^{\mathbf{p}_1}(\Lambda^2) \quad \text{and} \quad D_{\phi_1} : \ell_{(\omega_2)}^{\mathbf{p}_2}(\Lambda^2) \rightarrow M_{(\omega_2)}^{\mathbf{p}_2}(\mathbf{R}^d) \quad (3.5)$$

are continuous.

Furthermore, since

$$\frac{\omega_2(x, \xi)}{\omega_1(y, \eta)} \lesssim \omega_0(x, \eta, \xi - \eta, y - x),$$

it follows from (3.2) that

$$\frac{\omega_2(X)}{\omega_1(Y)} \lesssim \omega(X, Y), \quad X = (x, \xi) \in \mathbf{R}^{2d}, \quad Y = (y, \eta) \in \mathbf{R}^{2d}.$$

holds. Hence Theorem 2.3 shows that

$$A : \ell_{(\omega_1)}^{\mathbf{p}_1}(\Lambda^2) \rightarrow \ell_{(\omega_2)}^{\mathbf{p}_2}(\Lambda^2)$$

is continuous. Hence, if $\text{Op}(a)$ is defined by (3.3), it follows that $\text{Op}(a)$ from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}'_{1/2}(\mathbf{R}^d)$ extends to a continuous map from $M_{(\omega_1)}^{\mathbf{p}_1}(\mathbf{R}^d)$ to $M_{(\omega_2)}^{\mathbf{p}_2}(\mathbf{R}^d)$.

It remains to prove the uniqueness of the extension, If $\max(\mathbf{p}_1) < \infty$, then the uniqueness follows from the fact that $\mathcal{S}_{1/2}(\mathbf{R}^d)$ is dense in $M_{(\omega_1)}^{\mathbf{p}_1}(\mathbf{R}^d)$. If instead $p < \infty$, then $q < \infty$, and $\mathcal{S}_{1/2}(\mathbf{R}^{2d})$ is dense in $M_{(\omega_0)}^{p,q}(\mathbf{R}^{2d})$. The uniqueness now follows in this case from (3.1)', and the fact that $\text{Op}(a)$ is uniquely defined as an operator from $\mathcal{S}'_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}_{1/2}(\mathbf{R}^d)$, when $a \in \mathcal{S}_{1/2}(\mathbf{R}^{2d})$.

Finally assume that $p = \infty$ and $\max(\mathbf{p}_1) = \infty$. Then (2.8)' shows that $q \leq 1$. In particular, if $f \in M_{(\omega_1)}^{\mathbf{p}_1}(\mathbf{R}^d)$ then $f \in M_{(\omega_1)}^\infty(\mathbf{R}^d)$. The

uniqueness now follows from the fact that $\text{Op}(a)f$ is uniquely defined as an element in $M_{(\omega_2)}^\infty(\mathbf{R}^d)$, in view of Theorem A.2 in [50]. \square

We have also the following result on Schatten-von Neumann properties for pseudo-differential operators.

Theorem 3.4. *Let $t \in \mathbf{R}$, $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $\omega_0 \in \mathcal{P}_E(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ be such that*

$$\frac{\omega_2(x, \xi)}{\omega_1(y, \eta)} \asymp \omega_0((1-t)x + ty, t\xi + (1-t)\eta, \xi - \eta, y - x) \quad (3.6)$$

Also let $p, p_j, q, q_j \in (0, \infty]$ be such that

$$p_1 \leq p, \quad q_1 \leq \min(p, p'), \quad p_2 \geq \max(p, 1), \quad q_2 \geq \max(p, p').$$

Then

$$M_{(\omega_0)}^{p_1, q_1}(\mathbf{R}^{2d}) \subseteq s_{t,p}(\omega_1, \omega_2) \subseteq M_{(\omega_0)}^{p_2, q_2}(\mathbf{R}^{2d}) \quad (3.7)$$

and

$$\|a\|_{M_{(\omega_0)}^{p_2, q_2}} \lesssim \|a\|_{s_{t,p}(\omega_1, \omega_2)} \lesssim \|a\|_{M_{(\omega_0)}^{p_1, q_1}}. \quad (3.8)$$

Proof. We use the same notations as in the proof of Theorem 3.1. The result is true for $p \in [1, \infty]$ in view of Theorem A.3 in [50] and Proposition 1.1. Hence it suffices to prove the assertion for $p \in (0, 1)$.

By Proposition 1.7 in [51] and its proof, it suffices to prove the result for $t = 0$.

It follows from (1.19) and (3.3) that

$$\begin{aligned} \|\text{Op}(a)\|_{\mathcal{S}_p(\omega_1, \omega_2)} &= \|D_{\phi_1} \circ A \circ C_{\phi_1}\|_{\mathcal{S}_p(\omega_1, \omega_2)} \\ &\lesssim \|D_{\phi_1}\|_{\mathcal{S}_\infty(\ell_{(\omega_2)}^2(\Lambda^2), M_{(\omega_2)}^2)} \|A\|_{\mathcal{S}_p(\ell_{(\omega_1)}^2(\Lambda^2), \ell_{(\omega_2)}^2(\Lambda^2))} \|C_{\phi_2}\|_{\mathcal{S}_\infty(M_{(\omega_1)}^2, \ell_{(\omega_1)}^2(\Lambda^2))} \\ &\asymp \|A\|_{\mathcal{S}_p(\ell_{(\omega_1)}^2(\Lambda^2), \ell_{(\omega_2)}^2(\Lambda^2))} \lesssim \|A\|_{\mathbb{U}^{p,p}(\omega, \Lambda^2)} \asymp \|a\|_{M_{(\omega_0)}^{p,p}}, \end{aligned}$$

and the result follows. \square

Remark 3.5. Theorems 3.1 and 3.4 are related to certain results [35, 38] when the involved weights are trivial, and the involved Lebesgue exponents belong to the subset $[1, \infty]$ of $(0, \infty]$. More precisely, let $\mathbf{p} \in [1, \infty]^{4d}$ be given by

$$\mathbf{p} = (p_1, \dots, p_1, p_2, \dots, p_2, q_1, \dots, q_1, q_2, \dots, q_2),$$

and each p_j and q_j occur d times. Then S. Molahajloo and G. E. Pfander investigate in [35, 38], continuity of pseudo-differential operators with symbols in $M^{\mathbf{p}}(\mathbf{R}^{2d})$, when acting between $M^{r_1, s_1}(\mathbf{R}^d)$ and $M^{r_2, s_2}(\mathbf{R}^d)$, for some $r_j, s_j \in [1, \infty]$ (cf. Theorem 1.3 in [35]).

We note that there are some overlaps between Theorems 3.1 and 3.4 and the results in [35, 38]. On the other hand, the results in [35, 38], and Theorems 3.1 and 3.4 do not contain each others, since the assumptions on the symbols are more restrictive in Theorems 3.1 and 3.4, while

the assumptions on domains and image spaces are more restrictive in [35, 38].

Next we show that Theorem 3.4 is optimal with respect to p . More precisely, we have the following result

Theorem 3.6. *Let $t \in \mathbf{R}$, $\omega_k \in \mathcal{P}_E(\mathbf{R}^{2d})$, $k = 1, 2$, and $\omega_0 \in \mathcal{P}_E(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ be such that (3.6) holds. Also let $p, q, r \in (0, \infty]$, and suppose*

$$M_{(\omega_0)}^{p,q}(\mathbf{R}^{2d}) \subseteq s_{t,r}(\omega_1, \omega_2). \quad (3.9)$$

Then the following is true:

- (1) $p \leq r$ and $q \leq \min(2, r)$;
- (2) if in addition $\omega_1, \omega_2 \in \mathcal{P}(\mathbf{R}^{2d})$ and $r \geq 2$, then $q \leq p'$.

We need some preparations for the proof. The following result concerning Wigner distributions extends [50, Proposition A.4] to involve Lebesgue exponents smaller than one (cf. (1.16)). We omit the proof since the arguments are the same as in the proof of [50, Proposition A.4]. (See also [21, 48] and the references therein for related results.)

Proposition 3.7. *Let $t \in \mathbf{R}$, and let $p_j, q_j, p, q \in (0, \infty]$ be such that $p \leq p_j, q_j \leq q$, for $j = 1, 2$, and*

$$1/p_1 + 1/p_2 = 1/q_1 + 1/q_2 = 1/p + 1/q. \quad (3.10)$$

Also let $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $\omega \in \mathcal{P}_E(\mathbf{R}^{2d} \oplus \mathbf{R}^{2d})$ be such that

$$\omega_0((1-t)x + ty, t\xi + (1-t)\eta, \xi - \eta, y - x) \lesssim \omega_1(x, \xi)\omega_2(y, \eta). \quad (3.11)$$

Then the map $(f_1, f_2) \mapsto W_{f_1, f_2}^t$ from $\mathcal{S}'_{1/2}(\mathbf{R}^d) \times \mathcal{S}'_{1/2}(\mathbf{R}^d)$ to $\mathcal{S}'_{1/2}(\mathbf{R}^{2d})$ restricts to a continuous mapping from $M_{(\omega_1)}^{p_1, q_1}(\mathbf{R}^d) \times M_{(\omega_2)}^{p_2, q_2}(\mathbf{R}^d)$ to $M_{(\omega_0)}^{p, q}(\mathbf{R}^{2d})$, and

$$\|W_{f_1, f_2}^t\|_{M_{(\omega_0)}^{p, q}} \lesssim \|f_1\|_{M_{(\omega_1)}^{p_1, q_1}} \|f_2\|_{M_{(\omega_2)}^{p_2, q_2}} \quad (3.12)$$

when $f_1, f_2 \in \mathcal{S}'_{1/2}(\mathbf{R}^d)$.

We have now the following extension of Corollary 4.2 (1) in [45].

Corollary 3.8. *Let $p \in (0, \infty]$, $q \in (2, \infty]$, $t \in \mathbf{R}$, and let $\omega_2 \in \mathcal{P}_E(\mathbf{R}^{2d})$ and $\omega_0 \in \mathcal{P}_E(\mathbf{R}^{4d})$ be such that*

$$\omega_0((1-t)x, t\xi, \xi, -x) \lesssim \omega_2(x, \xi).$$

Then there is an element a in $M_{(\omega_0)}^{p, q}(\mathbf{R}^{2d})$ such that $\text{Op}_t(a)$ is not continuous from $\mathcal{S}_{1/2}(\mathbf{R}^d)$ to $M_{(\omega_2)}^{2, 2}(\mathbf{R}^d)$.

Proof. Let $a = W_{f_2, f_1}^t$, where $f_1 \in \Sigma_1(\mathbf{R}^d) \setminus 0$ and $f_2 \in M_{(\omega_2)}^{q, p}(\mathbf{R}^d) \setminus M_{(\omega_2)}^{2, 2}(\mathbf{R}^d)$. Such choices of f_2 are possible in view of Proposition 1.5.

By using the fact that ω_0 and ω_2 are moderate weights, it follows that (3.11) holds when $\omega_1(x, \xi) = e^{c(|x|+|\xi|)}$, and the constant $c > 0$ is chosen

large enough. By Proposition 1.1, it follows that $f_1 \in M_{(\omega_1)}^{p,q}(\mathbf{R}^d)$. Hence $a \in M_{(\omega_0)}^{p,q}(\mathbf{R}^{2d})$ in view of Proposition 3.7.

On the other hand, if $f \in \mathcal{S}_{1/2}(\mathbf{R}^d) \setminus 0$ is chosen such that f and f_1 are not orthogonal, then

$$\text{Op}_t(a)f = (f, f_1) \cdot f_2 \in M_{(\omega_2)}^{q,p}(\mathbf{R}^d) \setminus M_{(\omega_2)}^{2,2}(\mathbf{R}^{2d}),$$

and the result follows. \square

We also need the following lemma. We omit the proof since the result is a special case of Proposition 4.3 in [51]. Here \check{f} is defined as $\check{f}(x) = f(-x)$ when $f \in \mathcal{S}'_{1/2}(\mathbf{R}^d)$, and recall from Subsection 1.8 that $(\mathcal{F}_\sigma a)(X) = 2^d \hat{a}(-2\xi, 2x)$ when $X = (x, \xi) \in \mathbf{R}^{2d}$.

Lemma 3.9. *Let $\omega_1, \omega_2 \in \mathcal{P}_E(\mathbf{R}^{2d})$, $a \in \mathcal{S}'(\mathbf{R}^{2d})$, and that $p \in (0, \infty]$. Then*

$$\mathcal{F}_\sigma(s_p^w(\omega_1, \omega_2)) = s_p^w(\omega_1, \check{\omega}_2).$$

Proof of Theorem 3.6. We may assume that $t = 1/2$, and consider first the case when $1 \leq r$. Let $\mathcal{M}_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ and $\mathcal{W}_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ be the modulation spaces when the symplectic Fourier transform is used instead of the ordinary Fourier transform in the definition of modulation space norms of $M_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ and $W_{(\omega)}^{p,q}(\mathbf{R}^{2d})$, respectively. Then (3.9) is equivalent to

$$\text{Op}^w(\mathcal{M}_{(\omega)}^{p,q}) \subseteq \mathcal{S}_r(\omega_1, \omega_2), \quad \text{when} \quad \frac{\omega_2(X - Y)}{\omega_1(X + Y)} \asymp \omega(X, Y)$$

(see e.g. [50]). Let $\phi \in \Sigma(\mathbf{R}^{2d})$ and Λ in (1.6) be chosen such that $\{\phi(\cdot - \mathbf{j})e^{-2i\sigma(\cdot, \mathbf{k})}\}_{\mathbf{j}, \mathbf{k} \in \Lambda^2}$ is a Gabor frame. Also let $\vartheta(\mathbf{k}) = \omega(0, \mathbf{k})$, $c \in \ell_{(\vartheta)}^\infty(\Lambda^2)$, $c_0(0, \mathbf{k}) = c(\mathbf{k})$ and $c_0(\mathbf{j}, \mathbf{k}) = 0$ when $\mathbf{j} \neq 0$, and let

$$a(X) \equiv \sum_{\mathbf{k} \in \Lambda^2} c(\mathbf{k}) \phi(X) e^{-2i\sigma(X, \mathbf{k})} = \sum_{\mathbf{j}, \mathbf{k} \in \Lambda^2} c_0(\mathbf{j}, \mathbf{k}) \phi(X - \mathbf{j}) e^{-2i\sigma(X, \mathbf{k})}.$$

Then $a \in \mathcal{M}_{(\omega)}^{p,\infty}(\mathbf{R}^{2d})$ for every $p \in (0, \infty]$. Furthermore,

$$a \in \mathcal{M}_{(\omega)}^{p,q} \iff a \in \mathcal{W}_{(\omega)}^{p,q} \iff c \in \ell_{(\vartheta)}^q \quad (3.13)$$

holds for every $p, q \in (0, \infty]$.

Now if $q > r$, then choose $c \in \ell_{(\vartheta)}^q \setminus \ell_{(\vartheta)}^r$, and it follows from (3.7) and (3.13) that $a \in \mathcal{M}_{(\omega)}^{p,q} \setminus s_r^w(\omega_1, \omega_2)$. This shows that $q \leq r$ when (3.9) holds.

Assume instead that $p > r$, let $q \in (0, \infty]$ be arbitrary, choose $c \in \ell_{(\vartheta)}^p \setminus \ell_{(\vartheta)}^r$, and consider

$$b = \mathcal{F}_\sigma a \in \mathcal{F}_\sigma \mathcal{W}_{(\omega)}^{q,p} = \mathcal{M}_{(\omega_T)}^{p,q}, \quad \omega_T(X, Y) = \omega(Y, X).$$

By Lemma 3.9, (3.7) and (3.13) it follows that $b \notin s_r^w(\omega_1, \check{\omega}_2)$. This shows that $p \leq r$ when (3.9) holds, and the result follows in the case $r \geq 1$.

Next assume that $r < 1$. If (3.9) holds for some $q > r$, then it follows by (real) interpolation between the cases (3.9) and

$$\mathrm{Op}^2(\mathcal{M}_{(\omega)}^{2,2}(\mathbf{R}^{2d})) = \mathcal{I}_2(M_{(\omega_1)}^2(\mathbf{R}^d), M_{(\omega_2)}^2(\mathbf{R}^d))$$

that (3.9) holds for $r = 1$ and some $q > 1$. This contradicts the first part of the proof. If instead (3.9) holds for some $p > r$, then it again follows by interpolation that (3.9) holds for $r = 1$ and some $p > 1$, which again contradicts the first part of the proof. This shows that $p, q \leq r$ if (3.9) should hold. Furthermore, by Corollary (3.8) it follows that $q \leq 2$ when (3.9) holds, and (1) follows.

It remains to prove (2). By [25, Corollary 3.5] it follows that the result is true for trivial weights in the modulation space norms, and the result is carried over to the case with non-trivial weights by using lifting properties, established in [28]. The proof is complete. \square

4. APPLICATIONS TO THE HÖRMANDER-WEYL CALCULUS

In this section we apply the results in the previous section to deduce Schatten-von Neumann properties in the Hörmander-Weyl calculus. (See e. g. [33, 47], Sections 18.4–18.6 in [31] and Subsection 1.8 for approaches or notations.)

The following result is a consequence of Theorem 4.4 (1) in [47] in the case $p \geq 1$, while the latter result do not touch the case when $p < 1$.

Theorem 4.1. *Let $p \in (0, 2]$, g be feasible on W , and let m be a g -continuous weight on W such that $m \in L^p(W)$. Then $S(m, g) \subseteq s_p^w(W)$.*

We need some preparations for the proofs. First we recall that for any feasible metric g and any $X \in W$, there are symplectic coordinates, and numbers

$$0 < \lambda_d(X) \leq \cdots \leq \lambda_1(X) \leq 1$$

such that

$$g_X(Y) = \sum_{k=1}^d \lambda_k(X)(y_k^2 + \eta_k^2), \quad g_X^\sigma(Y) = \sum_{k=1}^d \lambda_k(X)^{-1}(y_k^2 + \eta_k^2),$$

where $Y = (y, \eta) \in W$ in these coordinates. The intermediate metric

$$g_X^0(Y) = \sum_{k=1}^d (y_k^2 + \eta_k^2)$$

is symplectically invariant defined and is *symplectic*, i. e. $(g^0)^\sigma = g^0$.

We have the following lemma.

Lemma 4.2. *Let $p \in (0, 1]$, \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces, and let $T_j \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $j \geq 1$. Then*

$$\|T\|_{\mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)} \leq \left(\sum_{k=1}^{\infty} \|T_k\|_{\mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)}^p \right)^{1/p}, \quad T = \sum_{k=1}^{\infty} T_k,$$

provided the right-hand side makes sense as an element in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.

We refer to [37, Appendix 1.1] for the proof of Lemma 4.2.

Corollary 4.3. *Let $p \in (0, 1]$, $t \in \mathbf{R}$ and $a \in \mathcal{S}'(\mathbf{R}^{2d})$. Also let $\{\varphi_j\}_{j \in I}$ be a sequence in $\mathcal{S}(\mathbf{R}^{2d})$ such that $0 \leq \varphi_j \leq 1$ for every j and*

$$\sum_{j \in I} \varphi_j = 1.$$

Then

$$\|a\|_{s_{t,p}(\mathbf{R}^{2d})} \leq \left(\sum_{j \in I} \|\varphi_j a\|_{s_{t,p}(\mathbf{R}^{2d})}^p \right)^{1/p}. \quad (4.1)$$

Proof. Let $a_j = \varphi_j a$, $I_k \subseteq I$, $k \geq 1$ be a sequence of increasing and finite sets such that $\bigcup_{k=1}^{\infty} I_k = I$, and set $b_k = \sum_{j \in I_k} a_j$. We may assume that

$$\sum_{j \in I} \|a_j\|_{s_{t,p}}^p < \infty, \quad (4.2)$$

since otherwise there is nothing to prove.

If $k_1 \leq k_2$, then Lemma 4.2 gives

$$\|b_{k_2} - b_{k_1}\|_{s_{t,p}}^p \leq \sum_{j \in I_{k_2} \setminus I_{k_1}} \|a_j\|_{s_{t,p}}^p \quad \text{and} \quad \|b_k\|_{s_{t,p}}^p \leq \sum_{j \in I_k} \|a_j\|_{s_{t,p}}^p. \quad (4.3)$$

By (4.2) we get $\|b_{k_2} - b_{k_1}\|_{s_{t,p}} \rightarrow 0$ as $k_1, k_2 \rightarrow \infty$, and by completeness, there is a unique element $b \in s_{t,p}$ such that $\|b_k - b\|_{s_{t,p}} \rightarrow 0$ as $k \rightarrow \infty$.

Since $s_{t,p} \subseteq s_{t,2} = L^2$, we get $b \in L^2$ and $b_k \rightarrow b$ in L^2 as well when $k \rightarrow \infty$. Furthermore, since $|b_k| \leq |a|$ and $b_k \rightarrow a$ pointwise, Beppo Levi's theorem gives that $a \in L^2$ and $a = b$. The result now follows by letting k tends to ∞ in (4.3). \square

Proof of Theorem 4.1. Since $S(m, g) \subseteq S(m, g^0)$ when g^0 is the symplectic metric of g , and that m is g^0 -continuous when m is g -continuous, we may assume that g is symplectic. (Cf. [47].)

Let $c > 0$, $C > 0$, I , X_j , U_j and φ_j be chosen as in Remarks 2.3 and 2.4 in [47], except that we may assume that $c > 0$ and $C > 0$ was chosen such that

$$C^{-1}m(X) \leq m(Y) \leq Cm(X) \quad \text{and} \quad C^{-1}g_X \leq g_Y \leq Cg_X$$

$$\text{when } g_Y(X - Y) < 2c$$

Also set $a_j = \varphi_j a$. Then $\text{supp } a_j \subseteq U_j$, and the set of a_j is bounded in $S(m, g)$. We first estimate $\|a_j\|_{s_p^w}$ for a fixed $j \in I$.

Let $B_r(X_0)$ denotes the open ball with center at X_0 and radius r , $\Phi \in C_0^\infty(B_1(0)) \setminus 0$ be such that $0 \leq \Phi \leq 1$, and choose symplectic coordinates $X = (x, \xi) \in \mathbf{R}^d \times \mathbf{R}^d \simeq \mathbf{R}^{2d}$ such that g_{X_j} takes the form

$$g_{X_j}(X) = g_{X_j}(x, \xi) = |X|^2 = |x|^2 + |\xi|^2,$$

in these coordinates. Then for every multi-index α , we have $\partial_g^\alpha = \partial_x^{\alpha_1} \partial_\xi^{\alpha_2}$ for some α_1 and α_2 such that $\alpha_1 + \alpha_2 = \alpha$, and the support of a_j is contained in $B_c(X_j)$. Let $N \geq 0$ be an integer such that $Np > d$. By Theorem 3.1 in [19] and Theorem 3.4 we get

$$\begin{aligned} \|a_j\|_{s_p^w}^p &\leq C_1 \iiint |V_\Phi a_j(x, \xi, \eta, y)|^p dx d\xi d\eta dy \\ &\leq C_2 \sum_{|\alpha+\beta|=2N} \iiint |V_{\partial^\beta \Phi}(\partial^\alpha a_j)(x, \xi, \eta, y) \langle (y, \eta) \rangle^{-2N}|^p dx d\xi d\eta dy \\ &\leq C_3 \sum_{|\alpha| \leq 2N} \iiint |V_\Phi(\partial^\alpha a_j)(x, \xi, \eta, y)|^p \langle (y, \eta) \rangle^{-2Np} dx d\xi d\eta dy, \end{aligned} \quad (4.4)$$

for some constants C_1, \dots, C_3 which are independent of $j \in I$.

We shall estimate the last integrand in (4.4). Let $\chi_{0,j}$ be the characteristic function of $U_j = B_c(X_j)$, and let χ_j be the characteristic function of $U_j = B_{1+c}(X_j)$. Then

$$|(\partial^\alpha a_j)(X)| \leq C_{1,\alpha} m(X) \chi_{0,j}(X) \leq C_{2,\alpha} m(X_j) \chi_{0,j}(X),$$

for some constants $C_{1,\alpha}$ and $C_{2,\alpha}$, which only depend on α . Here the last step follows from the fact that m is g -continuous. This gives

$$\begin{aligned} |V_\Phi(\partial^\alpha a_j)(x, \xi, \eta, y)| &\leq (2\pi)^{-d/2} \int_{\mathbf{R}^{2d}} |(\partial^\alpha a_j)(Z - X)| \Phi(Z) dZ \\ &\leq C_{3,\alpha} m(X_j) \int_{\mathbf{R}^{2d}} |\chi_{0,j}(Z - X)| \Phi(Z) dZ \leq C_{4,\alpha} m(X_j) \chi_j(X) \\ &= C_{5,\alpha} \left(m(X_j)^p \int \varphi_j(Z) dZ \right)^{1/p} \chi_j(X) \\ &\leq C_{6,\alpha} \left(\int m(Z)^p \varphi_j(Z) dZ \right)^{1/p} \chi_j(X), \end{aligned}$$

for some constants $C_{3,\alpha}, \dots, C_{6,\alpha}$, which only depend on α .

By combining the last estimate with (4.4), we get

$$\begin{aligned} \|a_j\|_{s_p^w}^p &\leq C_1 \left(\iint \chi_j(X)^p \langle Y \rangle^{-2Np} dX dY \right) \cdot \left(\int m(Z)^p \varphi_j(Z) dZ \right) \\ &\leq C_2 \int m(Z)^p \varphi_j(Z) dZ, \end{aligned}$$

for some constants C_1 and C_2 , which are independent of j .

A combination of the last estimate and Corollary 4.3 now gives

$$\|a\|_{s_p^w}^p \leq \sum_j \|a_j\|_{s_p^w}^p \lesssim \sum_j \int m(Z)^p \varphi_j(Z) dZ = \|m\|_{L^p}^p,$$

and the result follows. \square

By similar arguments as in the proof of [47, Theorem 2.11], Theorem 4.1 gives the following results involving suitable Sobolev type spaces, introduced in by Bony and Chemin in [7] (see also [33]). The details are left for the reader. Here recall that the Riemannian metric g on W is called *split* if there are symplectic coordinates such that

$$g_X(y, \eta) = g_X(y, -\eta), \quad X \in W, Y = (y, \eta) \in W,$$

in these coordinates.

Theorem 4.4. *Let $t = 1/2$, $a \in \mathcal{S}'(W)$, $p \in (0, \infty]$, g be strongly feasible on W , and let m, m_1, m_2 be g -continuous and (σ, g) -temperate weights on W such that $m_2 m / m_1 \in L^p(W)$. Then $S(m, g) \subseteq s_{t,p}(m_1, m_2, g)$.*

Furthermore, if in addition g is split, then $S(m, g) \subseteq s_{t,p}(m_1, m_2, g)$ holds for general $t \in \mathbf{R}$.

Remark 4.5. We note that [47, Theorem 2.11] covers Theorem 4.4 when $p \geq 1$. We also note that the assumption $a \in S(m, g)$ is missing, and that $s_{t,p}^w$ and $s_{t,\sharp}^w$ should be $s_{t,p}$ and $s_{t,\sharp}$, respectively, in (3) and (4) in [47, Theorem 2.11].

5. APPLICATIONS TO COMPACTLY SUPPORTED SCHATTEN-VON NEUMANN SYMBOLS

In this section we introduce a subset $s_{t,p}^q(\mathbf{R}^{2d})$ of $s_{t,p}(\mathbf{R}^{2d})$ when $p, q \in (0, \infty]$. We show that the set of compactly supported elements in $s_{t,p}^q(\mathbf{R}^{2d})$ with $q = \min(p, 1)$ and $q = 1$ is a subspace and superspace, respectively, of compactly supported elements in $\mathcal{F}L^p$. This result goes back to [43, 44] in the case $p \geq 1$ and $t = 1/2$. The proof is based on Theorem 3.4 and certain characterizations given here, which might be of independent interests.

First we make the following definition. Here ON_d is the set of all orthonormal sequences in $L^2(\mathbf{R}^d)$.

Definition 5.1. Let $t \in \mathbf{R}$ and $p, q \in (0, \infty]$. Then $s_{t,p}^q(\mathbf{R}^{2d})$ consists of all $a \in \mathcal{S}'(\mathbf{R}^{2d})$ of the form

$$a = \sum_{j=0}^{\infty} \lambda_j W_{f_j, g_j}^t,$$

for some non-negative and non-increasing sequence $\{\lambda_j\}_{j=0}^{\infty} \in \ell^p(\mathbf{N})$, and some $\{f_j\}_{j=0}^{\infty} \in \text{ON}_d$ and $\{g_j\}_{j=0}^{\infty} \in \text{ON}_d$ which at the same time

are bounded in $M^{2q}(\mathbf{R}^d)$. For any $a \in s_{t,p}^q(\mathbf{R}^{2d})$ we set

$$\|a\|_{s_{t,p}^q} \equiv \|\{\lambda_j \|f_j\|_{M^{2q_0}} \|g_j\|_{M^{2q_0}}\}_{j=0}^\infty\|_{\ell^p}, \quad q_0 = \min(1, q).$$

In the following lemma we present some basic facts for $s_{t,p}^q(\mathbf{R}^{2d})$.

Lemma 5.2. *Let $p, q, p_j, q_j \in (0, \infty]$, $j = 0, 1, 2$ be such that $p_1 \leq p_2$, $q_1 \leq q_2$ and $q_0 \geq 1$. Then the following is true:*

- (1) $s_{t,p}^{q_0}(\mathbf{R}^{2d}) = s_{t,p}(\mathbf{R}^{2d})$ and $\|a\|_{s_{t,p}^{q_0}} = \|a\|_{s_{t,p}}$ when $a \in s_{t,p}(\mathbf{R}^{2d})$;
- (2) $\mathcal{S}(\mathbf{R}^{2d}) \subseteq s_{t,p}^q(\mathbf{R}^{2d})$;
- (3) $s_{t,p_1}^{q_1}(\mathbf{R}^{2d}) \subseteq s_{t,p_2}^{q_2}(\mathbf{R}^{2d})$;
- (4) if in addition $t = 1/2$, then $a \in s_{t,p}^q(\mathbf{R}^{2d})$, if and only if $\mathcal{F}_\sigma a \in s_{t,p}^q(\mathbf{R}^{2d})$, and

$$\|\mathcal{F}_\sigma a\|_{s_{t,p}^q} = \|a\|_{s_{t,p}^q}.$$

Proof. The assertion (1) follows from the fact that $L^2(\mathbf{R}^d)$ is continuously embedded in $M^{2q_0}(\mathbf{R}^d)$, since $2q_0 \geq 2$, (2) follows from Proposition 5.6 below, and (3) is a straight-forward consequence of the definitions.

Finally, (4) follows from the facts that $\mathcal{F}_\sigma W_{f,g} = W_{\check{f},g}$ and that $\|\check{f}\|_{M^{2q}} \asymp \|f\|_{M^{2q}}$. This gives the result. \square

The following result characterizes the set of compactly supported elements in $s_{t,p}(\mathbf{R}^{2d})$.

Theorem 5.3. *Let $t \in \mathbf{R}$ and $p, q \in (0, \infty]$ be such that $r \leq q$. Then*

$$\begin{aligned} s_{t,q}^q(\mathbf{R}^{2d}) \bigcap \mathcal{E}'(\mathbf{R}^{2d}) &\subseteq \mathcal{F}L^q(\mathbf{R}^{2d}) \bigcap \mathcal{E}'(\mathbf{R}^{2d}) \\ &= M^{p,q}(\mathbf{R}^{2d}) \bigcap \mathcal{E}'(\mathbf{R}^{2d}) \subseteq s_{t,q}(\mathbf{R}^{2d}) \bigcap \mathcal{E}'(\mathbf{R}^{2d}). \end{aligned} \quad (5.1)$$

We need some preparations for the proof. First we recall the following facts for the harmonic oscillator $H = H_d = |x|^2 - \Delta$ on $\mathcal{S}'(\mathbf{R}^d)$. We omit the proof since the result is a special case of [6, Theorem 3.5].

Lemma 5.4. *Let $p, q \in [1, \infty]$ and let $\omega, \vartheta_N \in \mathcal{P}(\mathbf{R}^{2d})$ be such that $\vartheta_N(x, \xi) = \langle (x, \xi) \rangle^N$, where N is an integer. Then H_d^N is a homeomorphism on $\mathcal{S}(\mathbf{R}^d)$, on $\mathcal{S}'(\mathbf{R}^d)$, and from $M_{(\vartheta_{2N}\omega)}^{p,q}(\mathbf{R}^d)$ to $M_{(\omega)}^{p,q}(\mathbf{R}^d)$.*

The next result concerns Schwartz kernels of linear operators.

Proposition 5.5. *Let T be a linear and continuous operator from $L^2(\mathbf{R}^{d_1})$ to $L^2(\mathbf{R}^{d_2})$ and such that the kernels of $T^* \circ T$ and $T \circ T^*$ belong to $\mathcal{S}(\mathbf{R}^{2d_1})$ and $\mathcal{S}(\mathbf{R}^{2d_2})$, respectively. Then the kernel of T belongs to $\mathcal{S}(\mathbf{R}^{d_2+d_1})$.*

The result should be available in the literature. In order to be self-contained we here present a proof, obtained in collaboration with A. Holst at Lund University, Sweden.

Proof. Let $N \in \mathbf{N}$, $\omega_{j,r}(x, \xi) = \langle (x, \xi) \rangle^r$ when $x, \xi \in \mathbf{R}^{d_j}$ and $r \in \mathbf{R}$, and let $f_j \in \mathcal{S}(\mathbf{R}^{d_j})$, $j = 1, 2$. Then $H^N \circ S \circ H^N$ is an operator with kernel in \mathcal{S} , if and only if S is an operator with kernel in \mathcal{S} .

By the assumptions and Lemma 5.4 we obtain

$$\|Tf_1\|_{L^2(\mathbf{R}^{d_2})}^2 = ((T^* \circ T)f_1, f_1)_{L^2(\mathbf{R}^{d_1})} \lesssim \|f_1\|_{M_{(\omega_1, -2N)}^2}^2$$

and

$$\|T^*f_2\|_{L^2(\mathbf{R}^{d_1})}^2 = ((T \circ T^*)f_2, f_2)_{L^2(\mathbf{R}^{d_2})} \lesssim \|f_2\|_{M_{(\omega_2, -2N)}^2}^2.$$

Hence

$$T \in \mathcal{B}(M_{(\omega_1, -2N)}^2(\mathbf{R}^{d_1}), L^2(\mathbf{R}^{d_2})) \quad \text{and} \quad T^* \in \mathcal{B}(M_{(\omega_2, -2N)}^2(\mathbf{R}^{d_2}), L^2(\mathbf{R}^{d_1})).$$

By duality it also follows that $T \in \mathcal{B}(L^2(\mathbf{R}^{d_1}), M_{(\omega_2, 2r)}^2(\mathbf{R}^{d_2}))$, since the dual of $M_{(\omega_{j,-N})}^2(\mathbf{R}^{d_j})$ equals $M_{(\omega_{j,N})}^2(\mathbf{R}^{d_j})$ when the L^2 form is used (cf. e. g. [21, Theorem 11.3.6]).

By interpolation between these results we get

$$T \in \mathcal{B}(M_{(\omega_1, -N)}^2(\mathbf{R}^{d_1}), M_{(\omega_2, N)}^2(\mathbf{R}^{d_2})).$$

and since

$$\bigcap_{r \in \mathbf{R}} M_{(\omega_{j,N})}^2(\mathbf{R}^{d_j}) = \mathcal{S}(\mathbf{R}^{d_j}) \quad \text{and} \quad \bigcup_{r \in \mathbf{R}} M_{(\omega_{j,r})}^2(\mathbf{R}^{d_j}) = \mathcal{S}'(\mathbf{R}^{d_j})$$

when $j = 1, 2$ also in topological sense in view of e. g. [46], we obtain $T \in \mathcal{B}(\mathcal{S}'(\mathbf{R}^{d_1}), \mathcal{S}(\mathbf{R}^{d_2}))$. This implies that the kernel of T belongs to $\mathcal{S}(\mathbf{R}^{d_2+d_1})$. \square

The next result extends in several ways Lemma 4.1.2 in [43] and concerns suitable Wigner distribution expansions (see (1.16)).

Proposition 5.6. *Let $a \in \mathcal{S}(\mathbf{R}^{2d})$, $t \in \mathbf{R}$, $p, q \in (0, \infty]$ and let $H = |x|^2 - \Delta$ be the harmonic oscillator on \mathbf{R}^d . Then*

$$a = \sum_{j=0}^{\infty} \lambda_j W_{f_j, g_j}^t, \tag{5.2}$$

for some non-negative and non-increasing $\{\lambda_j\}_{j=0}^{\infty} \subseteq \mathbf{R}$, $\{f_j\}_{j=0}^{\infty} \in \text{ON}(L^2(\mathbf{R}^d))$ and $\{g_j\}_{j=0}^{\infty} \in \text{ON}(L^2(\mathbf{R}^d))$ such that

$$\lambda_j \geq 0, \quad f_j, g_j \in \mathcal{S}(\mathbf{R}^d), \quad j \geq 0,$$

and

$$\sum_{j=0}^{\infty} \lambda_j^p \|H^N f_j\|_{M^q} \|H^N g_j\|_{M^q} < \infty, \quad \text{when } N \geq 0. \tag{5.3}$$

Proof. By the spectral theorem of compact operators, (5.2) holds true for some non-negative and non-increasing sequence $\{\lambda_j\}_{j=0}^\infty$, and some $\{f_j\}_{j=0}^\infty$ and $\{g_j\}_{j=0}^\infty$ in ON_d . Since

$$e^{i(t-1/2)\langle D_\xi, D_x \rangle} W_{f,g}^t = W_{f,g}$$

and $e^{i(t-1/2)\langle D_\xi, D_x \rangle}$ is continuous on $\mathcal{S}(\mathbf{R}^{2d})$, it suffices to consider the Weyl case, $t = 1/2$.

First we assume that $T \equiv \text{Op}^w(a) \geq 0$, giving that (5.2) holds with $g_j = f_j$. The result is true for $p = 1$ and $q = 2$ in view of [43, Lemma 4.1.2]. Since the kernel of T belongs to \mathcal{S} , Proposition 5.5 shows that the kernel of $T_N = T^{1/2N}$ belongs to \mathcal{S} for every $N \geq 1$. Furthermore, if a_N is the Weyl symbol of T_N , then $a_N \in \mathcal{S}(\mathbf{R}^{2d})$, and by straightforward computations we get

$$a_N = \sum_j \lambda_j^{1/2N} W_{f_j, f_j}.$$

By [43, Lemma 4.1.2] we get

$$\sum_j \lambda_j^{1/2N} \|H^N f_j\|_{L^2}^2 < \infty$$

for every $N \geq 0$, and the result follows in the case $p > 0$ and $q = 2$.

Next assume that $q \in (0, 2)$, and let $\omega_r = \langle \cdot \rangle^r \in \mathcal{P}(\mathbf{R}^{2d})$, $q_0 = (2q)/(2-q)$ and $N_0 > dq_0$ be an integer. Then $\omega_{-2N_0} \in L^{q_0}(\mathbf{R}^{2d})$, and Hölder's inequality gives

$$\|H^N f\|_{M^q} \lesssim \|\omega_{-2N_0}\|_{L^{q_0}} \|H^N f\|_{M^2_{(\omega_{2N_0})}} \asymp \|H^{N+N_0} f\|_{L^2},$$

when $f \in \mathcal{S}(\mathbf{R}^d)$, and the result follows in this case from the case $q = 2$.

If instead $q \geq 2$, then $\|H^N f\|_{M^q} \lesssim \|H^N f\|_{L^2}$ for every admissible f , and the result again follows from the case when $q = 2$.

The assertion therefore follows if in the case $\text{Op}_t(a) \geq 0$.

Finally assume that $a \in \mathcal{S}(\mathbf{R}^{2d})$ is general. The Weyl symbol of the operators $T^* \circ T$ and $T \circ T^*$ are given by

$$b = \sum \lambda_j^2 W_{f_j, f_j} \quad \text{and} \quad c = \sum \lambda_j^2 W_{g_j, g_j},$$

respectively, and belong to $\mathcal{S}(\mathbf{R}^{2d})$, in view of Proposition 5.5. Since $T^* \circ T$ and $T \circ T^*$ are positive semi-definite, it follows from the first part of the proof that

$$\sum_j \lambda_j^p \|H^N f_j\|_{M^q}^2 < \infty \quad \text{and} \quad \sum_j \lambda_j^p \|H^N g_j\|_{M^q}^2 < \infty$$

hold for every $p, q \in (0, \infty]$ and $N \geq 0$. The estimate (5.3) now follows from these estimates and Cauchy-Schwartz inequality. \square

We also need the following result related to Theorem 3.1 in [19].

Lemma 5.7. *Let $p \in (0, 2]$ and let $\omega, \omega_1, \omega_2 \in \mathcal{P}_E(\mathbf{R}^{2d})$ be such that*

$$\omega(X_1 - X_2) \lesssim \omega_1(X_1)\omega_2(X_2), \quad X_1, X_2 \in \mathbf{R}^{2d}.$$

Then the map $(f, \phi) \mapsto V_\phi f$ is continuous from $M_{(\omega_1)}^p(\mathbf{R}^d) \times M_{(\omega_2)}^p(\mathbf{R}^d)$ to $L_{(\omega)}^p(\mathbf{R}^{2d})$, and

$$\|V_\phi f\|_{L_{(\omega)}^p} \leq C \|f\|_{M_{(\omega_1)}^p} \|\phi\|_{M_{(\omega_2)}^p}, \quad (5.4)$$

where the constant C is independent of $f \in M_{(\omega_1)}^p(\mathbf{R}^d)$ and $\phi \in M_{(\omega_2)}^p(\mathbf{R}^d)$.

Proof. First assume that $p \leq 1$. Let $\psi \in \Sigma_1(\mathbf{R}^d)$, $\varepsilon > 0$, $\Lambda = \varepsilon \mathbf{Z}^{2d}$, $\{c(\mathbf{j})\}_{\mathbf{j} \in \Lambda} \in \ell_{(\omega_1)}^p(\Lambda)$ and $\{d(\mathbf{k})\}_{\mathbf{k} \in \Lambda} \in \ell_{(\omega_2)}^p(\Lambda)$ be chosen such that

$$f(x) = \sum_{\mathbf{j}, \iota \in \varepsilon \mathbf{Z}^d} c(\mathbf{j}, \iota) \psi(x - \mathbf{j}) e^{i\langle x, \iota \rangle} \quad \text{and} \quad \phi(x) = \sum_{\mathbf{k}, \kappa \in \varepsilon \mathbf{Z}^d} d(\mathbf{k}, \kappa) \psi(x - \mathbf{k}) e^{i\langle x, \kappa \rangle}$$

(cf. Proposition 1.5).

By straight-forward computations it follows that

$$V_\phi f(X) = \sum_{\mathbf{j}, \mathbf{k} \in \Lambda} c(\mathbf{j}) \overline{d(\mathbf{k})} \Psi(X + \mathbf{k} - \mathbf{j}) R_{\mathbf{j}, \mathbf{k}}(X),$$

where $\Psi = V_\psi \psi \in \Sigma_1(\mathbf{R}^{2d})$ and $R_{\mathbf{j}, \mathbf{k}}$ is a function of exponential type such that $|R_{\mathbf{j}, \mathbf{k}}| = 1$, for every \mathbf{j} and \mathbf{k} . This gives

$$\begin{aligned} \|V_\phi f\|_{L_{(\omega)}^p}^p &\leq \sum_{\mathbf{j}, \mathbf{k} \in \Lambda} |c(\mathbf{j})|^p |d(\mathbf{k})|^p \|\Psi(\cdot + \mathbf{k} - \mathbf{j}) \omega\|_{L^p}^p \\ &\leq \sum_{\mathbf{j}, \mathbf{k} \in \Lambda} |c(\mathbf{j})|^p |d(\mathbf{k})|^p \|\Psi v\|_{L^p}^p \omega(\mathbf{j} - \mathbf{k})^p \\ &\lesssim \sum_{\mathbf{j}, \mathbf{k} \in \Lambda} |c(\mathbf{j}) \omega_1(\mathbf{j})|^p |d(\mathbf{k}) \omega_2(\mathbf{j})|^p \asymp \|f\|_{M_{(\omega_1)}^p} \|\phi\|_{M_{(\omega_2)}^p}, \end{aligned}$$

when $v \in \mathcal{P}_E(\mathbf{R}^{2d})$ is chosen such that ω is v -moderate. Here the first inequality follows from the fact that $p \leq 1$ and the last inequality follows from the assumptions. This gives the result in the case $p \leq 1$.

A slight modification of the proof of (2.8) in [50] gives the result in the remaining case where $p \in [1, 2]$. The details are left for the reader. \square

The next result concerns extensions of certain convolution relations in [43, 44] between Schatten-von Neumann symbols and Lebesgue spaces to the case when the Lebesgue parameters are allowed to be smaller than 1.

Proposition 5.8. *Let $p \in (0, 1]$ and $t = 1/2$. Then the map $(a, b) \mapsto a * b$ is continuous from $s_{t,p}^p(\mathbf{R}^{2d}) \times s_{t,p}^p(\mathbf{R}^{2d})$ to $L^p(\mathbf{R}^{2d})$.*

Proof. Let $a, b \in s_{t,p}^p$. Since $s_{t,p}^p \subseteq s_{t,1}$, it follows that $a * b$ is well-defined and belongs to L^1 , in view of Theorem 2.1 in [44].

Now let

$$\{\lambda_j\}_{j=0}^\infty \in \ell^p(\mathbf{N}), \quad \{\mu_j\}_{j=0}^\infty \in \ell^p(\mathbf{N}), \quad \{f_{l,j}\}_{j=0}^\infty \in \text{ON}(L^2(\mathbf{R}^d))$$

and

$$\{g_{l,j}\}_{j=0}^\infty \in \text{ON}(L^2(\mathbf{R}^d)), \quad l = 1, 2,$$

be such that $\lambda_j \geq 0$ and $\mu_j \geq 0$ for every $j \geq 0$,

$$\sup_{j,l} \|f_{l,j}\|_{M^{2p}} < \infty \quad \text{and} \quad \sup_{j,l} \|g_{l,j}\|_{M^{2p}} < \infty,$$

and

$$a = \sum_{j=0}^\infty \lambda_j W_{f_{1,j}, f_{2,j}} \quad \text{and} \quad b = \sum_{j=0}^\infty \mu_j W_{g_{1,j}, g_{2,j}}.$$

Then

$$\begin{aligned} \|a * b\|_{L^p}^p &= \int \left| \sum_{j,k} \lambda_j \mu_k (W_{f_{1,j}, f_{2,j}} * W_{g_{1,k}, g_{2,k}})(X) \right|^p dX \\ &\leq \sum_{j,k} \lambda_j^p \mu_k^p \|W_{f_{1,j}, f_{2,j}} * W_{g_{1,k}, g_{2,k}}\|_{L^p}^p. \end{aligned} \quad (5.5)$$

By straight-forward computations we get

$$|(W_{f_{1,j}, f_{2,j}} * W_{g_{1,k}, g_{2,k}})(x, \xi)| = C |(V_{\tilde{f}_{2,j}} g_{1,k})(x, \xi) (V_{\tilde{f}_{1,j}} g_{2,k})(x, \xi)|,$$

for some constant C . Hence Cauchy-Schwartz inequality and Lemma 5.7 give

$$\begin{aligned} \|W_{f_{1,j}, f_{2,j}} * W_{g_{1,k}, g_{2,k}}\|_{L^p} &= \|V_{\tilde{f}_{2,j}} g_{1,k} \cdot V_{\tilde{f}_{1,j}} g_{2,k}\|_{L^p} \\ &\leq \|V_{\tilde{f}_{2,j}} g_{1,k}\|_{L^{2p}} \|V_{\tilde{f}_{1,j}} g_{2,k}\|_{L^{2p}} \lesssim \|f_{1,j}\|_{M^{2p}} \|f_{2,j}\|_{M^{2p}} \|g_{1,k}\|_{M^{2p}} \|g_{2,k}\|_{M^{2p}}. \end{aligned}$$

By inserting this into (5.5) we get

$$\begin{aligned} \|a * b\|_{L^p} &\lesssim \left(\sum_{j,k} \lambda_j^p \mu_k^p \|f_{1,j}\|_{M^{2p}}^p \|f_{2,j}\|_{M^{2p}}^p \|g_{1,k}\|_{M^{2p}}^p \|g_{2,k}\|_{M^{2p}}^p \right)^{1/p} \\ &= \|a\|_{s_p^p} \|b\|_{s_p^p}, \end{aligned}$$

and the result follows. \square

Proof of Theorem 5.3. The equality and the last embedding in (5.1) follow from [51, Proposition 4.3] and Theorem 3.4. The first embedding in (5.1) follows from Corollary 2.12 in [44] in the case $p \geq 1$. It remains to prove the first embedding in (5.1) in the case $p < 1$.

Therefore, assume that $p < 1$, let $a \in \mathcal{E}'(\mathbf{R}^{2d})$ and choose $\varphi \in C_0^\infty(\mathbf{R}^{2d})$ such that $\varphi = 1$ on $\text{supp } a$. Then $\mathcal{F}_\sigma \varphi \in s_{t,p}^p(\mathbf{R}^{2d})$ by Proposition 5.6. Hence Lemma 5.2 and Proposition 5.8 give

$$\begin{aligned} \|a\|_{\mathcal{F}L^p} &= \|\varphi a\|_{\mathcal{F}L^p} \asymp \|(\mathcal{F}_\sigma \varphi) * (\mathcal{F}_\sigma a)\|_{L^p} \\ &\lesssim \|\mathcal{F}_\sigma \varphi\|_{s_{t,p}^p} \|\mathcal{F}_\sigma a\|_{s_{t,p}^p} \lesssim \|a\|_{s_{t,p}^p}, \end{aligned}$$

which gives the result. \square

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